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DYNAMICS OF TOWED UNDERWATER VEHICLES

Subproject No. SF 011 02 12 • Task No. 2563

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ABSTRACT

This report presents the analysis of a system which includes a maneuvering ship towing an underwater vehicle at the end of a long flexible cable. The equations of motion for both the cable and the underwater vehicle are also presented.

The cable is imagined to consist of many interconnected short rigid segments. The equations of motion for the system are formulated twice on the basis of two hypotheses: first for a simple hypothesis regarding the inertia of an accelerated body in a fluid, and secondly, for a more complete and a more accurate hydrodynamic hypothesis.

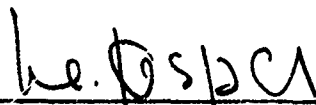
ADMINISTRATIVE INFORMATION

Minehunting Subproject PENTA (SF 011 02 12 Task 2563) was established by Bureau of Ships letter C-A11/NS 860 100 (531G), serial 531G-0736 of 30 January 1958, for the purpose of assisting in the finalization of a magnetic detector mine location system developed under bureau of ships Contract NObs-72083.

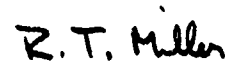
The objective of Project PENTA was redefined by Bureau of Ships letter C-F011 02-12 serial 631G-022 of 13 February 1961 to include vehicle aspects of mine location systems on a continuing basis.

The analysis was carried out while the first author was employed at U. S. Navy Mine Defense Laboratory during the summer of 1962 and subsequent brief periods by a personal service contract.

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1. In building towed underwater vehicles, design information, such as the stresses on fin axes, is needed. For this purpose, and also to evaluate the performance of potential on existing vehicles, a theoretical analysis is desirable. The purpose of this report is to derive a sufficient basis for this analysis. A study of the feasibility of computations is being made.
2. This work was carried out under subproject PENTA (SF 011 02 12 - Task 2563) as redefined by Bureau of Ships letter C-F 011 02 12 serial 631G-022 of 13 February 1961.
3. This report is presented to properly document the information contained herein and to distribute it to interested activities.

R.T. Miller

R. T. MILLER

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LIST OF SYMBOLS

.	the dot over a quantity denotes differentiation with respect to t , i.e., d/dt .
\bar{x}_λ or $x_{\lambda 1}$	two notations are used to denote vectors: either the vector bar over the letter, or a subscript to denote the component. For example $\bar{x}_\lambda = (x_\lambda, y_\lambda, z_\lambda) = x_{\lambda 1}$.
$\alpha, \beta, \gamma, \rho$	these Greek letters have specific meanings assigned further on in this section. They also are dummy variables. The quantity ρ is also the mass density in Appendix E.
$\sin(\bar{v}_1, \bar{v}_2)$	the sine of the angle between the vectors of \bar{v}_1 and \bar{v}_2 .
a_λ^*	a computational abbreviation for $\frac{1}{2}(\frac{w_\lambda d_\lambda}{g} + e_\lambda)$, where $\lambda = 1, 2, \dots, v$.
a_{v+1}^*	a computational abbreviation for $\frac{w_{v+1}}{g} (d_{v+1} + d_{v+5}) (\frac{d_{v+1} - d_{v+2}}{d_{v+1}}) + (e_{v+1}/2)$
a_{v+2}^*	A computational abbreviation for $\frac{w_{v+1}}{g} (d_{v+1} + d_{v+5}) \frac{d_{v+2}}{d_{v+1}} + (e_{v+1}/2)$
$AM_{\lambda i}$	component of moment on the λ th element in the i th direction due to the acceleration of the fluid.
b_λ	the buoyant force per unit length of the λ th segment due to the displaced water.
$B_{\lambda i}$	component of buoyant force on the λ th segment of cable.
BM_i	component of moment about the c.g. of the vehicle due to the buoyant force acting on it.
$b_{\lambda i j}$	a computational abbreviation for $e_\lambda s_{\lambda i} s_{\lambda j}$.

c_{ij}	the coefficients of the linear orthogonal coordinate transformation connecting fixed inertial coordinates with coordinates fixed in the towed vehicle with their origin at its c.g.
C_1	the proportionality constant between the square of the velocity of a cable segment and the tangential force per unit distance due to the velocity of the water.
$C_{1,v+1}$	for the vehicle is the same quantity as C_1 is for a cable segment.
C_2	the proportionality constant between the square of the velocity of a cable segment and the normal force per unit distance on the segment due to the velocity of the water.
$C_{2,v+1}$	for the vehicle is the same quantity as C_2 is for a cable segment.
C_3	the proportionality constant between the lift on the vehicle due to the fins in the vertical plane, i.e. the horizontal deflecting force, and the square of the velocity through the water times the function $F_{LVF}(\alpha - \alpha_1)$.
C_4	the corresponding constant for the drag on the fins in the vertical plane.
C_5	the proportionality constant between the lift due to the horizontal fins and the square of the velocity through the water times $F_{LHF}(\beta - \beta_1)$.
C_6	the corresponding constant for the drag of the fins in the horizontal plane.
$C1_{ij}$	added masses μ_{ij} , $i \leq j \leq 3$ (Reference 3).
$C2_{ij}$	added masses $\mu_{i+3,j+3}$, $i + 3 > 3$, $j + 3 > 3$.
$C3_{ij}$	added masses $\mu_{i,j+3}$, $i < 3$, $j + 3 > 3$; or $\mu_{i+3,j}$, $i + 3 > 3$, $j \leq 3$.
c.b., c.g., c.m.	are centers of buoyancy, gravity, mass.

d_λ	is the length of the λ th segment, for $\lambda = 1, 2, \dots, v$.
d_{v+1}	distances measured on the vehicle as indicated in Figure A3, where $i = 1, 2, \dots, 6$.
e_{ijk}	the skew symmetric three index symbol.
e_λ	the proportionality constant (Reference 1) between the normal component of acceleration of a cable segment and the force on it due to the accelerated motion of the fluid.
e'	equals e divided by the mass of the cable element.
$F_{\lambda i}$	the total force applied to the λ th element.
$F_{\lambda i}^*$	the total force applied to the λ th element exclusive of the mechanical forces at either end.
$F_{A\lambda i}$	the sum of all noninertial forces on the λ th element which depend on the acceleration of the element.
$F_{V\lambda i}$	the sum of all forces on the λ th element exclusive of those on the fins which depend on the velocity of the element but not on its acceleration.
$F_{G\lambda i}$	the sum of all forces on all the λ th element which depend on neither the element's acceleration nor velocity.
$(FIN)_i$	the force exerted by the fluid on the vehicle through the vehicle fins.
$(FINM)_i$	moment exerted by the fluid on the vehicle through the vehicle fins.
$(F_{app})_{\lambda i}$	that approximation to the force on the λ th element due to the velocity of the fluid obtained by neglecting the rotation of the λ th element.
$(F_{acc})_{\lambda i}$	the force on the λ th element due to the velocity of the fluid when the rotation of the element is taken into account.
$F_{DHF}(\beta - \beta_1)$	an empirical function giving the lift of the horizontal fins when the vehicle is parallel to the stream whose velocity is unity.

$F_{LVF}(\alpha - \alpha_1)$	an empirical function giving the lift on the vertical fins when the vehicle is parallel to a stream whose velocity is unity.
g	gravitational acceleration.
\bar{G}_λ	a computational abbreviation for $e_\lambda \bar{s}_\lambda (-d_\lambda \dot{s}_\lambda^2 / 2) + (w_\lambda - b_\lambda) d_\lambda \bar{k} + C_1 d_\lambda \bar{u}_\lambda^2 \bar{s}_\lambda$ $+ C_2 d_\lambda (\bar{s}_\lambda \times \bar{u}_\lambda) \bar{q}_\lambda + \overline{FIN}$, where \overline{FIN} is the zero vector unless $\lambda = v + 1$.
$H_{\lambda i}$	is the angular momentum of the λ th element about its c.g., where $\lambda = 1, 2, \dots, v + 1$.
I_{ij}^λ	equals $\Sigma(\Delta m) x_i x_j$, where (Δm) has an i th coordinate distance x_i from the c.g. of the element, and summation is carried out for all Δm into which the λ th element is decomposed.
I_{ij}^{v+1}	are the products of inertia of the vehicle about its c.g. referred to a frame of reference fixed in the vehicle.
\bar{i}	in Appendix E, a unit vector along the fixed x_1 axis.
\bar{j}	in Appendix E, a unit vector along the fixed x_2 axis.
\bar{k}	in Appendix E, a unit vector along the fixed x_3 axis.
K	in Appendix D, a proportionality constant which cancels out.
$\bar{\ell}, \bar{m}, \bar{n}$	unit vectors along ξ_1, ξ_2, ξ_3 axes respectively.
m_λ	the mass of the λ th element.
$M_{\lambda i}$	the sum of all the moments acting on the λ th element, about the i th axis.
$M_{\lambda i}^*$	the sum of all the moments on the λ th element about the i th axis which depend on neither the element's velocity nor its acceleration.

$M_{V\lambda i}$

the sum of all the moments on the λ th element about the i th axis which depend on the element's velocity but not on its acceleration.

$M_{A\lambda i}$

the sum of all of the moments on the λ th element which depend in any way on the acceleration of the element.

\bar{n}

a unit vector along the ξ_3 axis fixed in the vehicle.

q_λ

a unit vector along the intersection of the plane normal to the λ th element and the plane containing two lines, one along the axis of the element and the other in the direction of the velocity of the fluid past the element; where $\lambda = 1, 2, \dots, v+1$.

\bar{q}_λ

unit vectors related to the direction of flow of fluid past the vehicle fins, where $\lambda = v+2, v+3, v+4, v+5$.

\bar{r}

a general displacement vector used in Appendix E to relate the Kirchhoff force to an inertial frame.

\bar{s}_λ

a unit vector along the λ th element, $\lambda = 1, 2, \dots, v+1$.

\bar{s}_0

for the towing vehicle, a unit vector corresponding to \bar{s}_λ .

\bar{T}_λ

the mechanical force exerted on the λ th element by the $\lambda - 1$ element.

\bar{T}_{v+2}

the negative of the thrust, such as from a propeller, that the vehicle exerts on the water.

\bar{u}_λ

the velocity of the c.g. of the λ th element.

u_i

components of linear and angular velocity of the vehicle referred to axes fixed in the vehicle, where $i = 1, 2, \dots, 6$.

\bar{v}_{vf}

the velocity through the fluid of the vertical fins.

\bar{v}_{vfp}

the projection of \bar{v}_{vf} on the $\xi_1 \xi_2$ plane.

\bar{v}_{hf}

the velocity through the fluid of the horizontal fins.

\bar{v}_{hfp}

the projection of \bar{v}_{hf} on the $\xi_1 \xi_3$ plane.

w_λ

weight per unit length of the λ th element in a vacuum.

\bar{x}_λ

a radius vector from an arbitrary inertial origin to that terminal end of the λ th element nearest the towing vehicle.

\bar{x}_G	the radius vector from an arbitrary inertial origin to the c.g. of the vehicle.
$\bar{x}_{v/s}$	the vector from the towing vehicle to the towed vehicle.
\bar{x}	in Appendix A, the vector from the c.g. of the vehicle to the fin axis.
α	the angular deflection of the vertical fin relative to the vehicle.
α_1	the angle between the vertical fin and the projection on the $\xi_1 \xi_2$ plane of its velocity through the water.
β	the angular deflection of the horizontal fin relative to the vehicle.
β_1	the angle between the horizontal fin and the projection on the $\xi_1 \xi_3$ plane of its velocity through the water.
δ_{ij}, δ_j^i	the Kronecker delta. It is zero unless $i = j$ in which case it is one.
ξ, η, ζ	coordinates fixed in the vehicle with their origin at its c.g.
λ	the index denoting the cable segment. $\lambda = v + 1$ denotes the vehicle.
μ	also the index denoting the cable segment.
μ_{ij}	the surface integral $\rho \int \varphi_i (\partial \varphi_h / \partial n) dS$, φ_i is the velocity potential due to motion with unit velocity in the i th direction.
v	the number of segments into which the cable is divided.
ρ	the mass density of the fluid.
φ_λ	the angle between \bar{r}_λ and \bar{u}_λ .
$\varphi_{\lambda i}$	a computational abbreviation for $B_{\lambda i} + FD_{\lambda i} + FL_{\lambda i} + (FIN)_i$, where $(FIN)_i$ is zero unless $\lambda = v + 1$.

$\chi(\bar{u}_\lambda)^2$

the factor relating the velocity through the fluid of the λ th element to the normal force caused by this velocity.

$\psi(\varphi_\lambda)$

the factor relating the orientation of the λ th element and the normal force on it due to its velocity through the water.

$\bar{\omega}_\lambda$

the angular velocity of the λ th element.

STATEMENT OF THE PROBLEM

An underwater vehicle is towed with a flexible cable by a maneuvering ship. A command is given which actuates the bow planes of the vehicle through a prescribed angle. The vehicle then changes its orientation and position. It is required to determine the position of the towed vehicle, and the cable tensions as time varies.

The only kinematic boundary condition imposed on the system consisting of the cable and underwater vehicle is that the end of cable attached to the towing platform follows a known curvilinear path along which the components of velocity and acceleration are given. It is assumed that the end of cable attached to the vehicle if extended, would intersect the longitudinal axis of the vehicle. This occurs, for example, when the bail is an arc of a circle whose center is on the longitudinal axis.

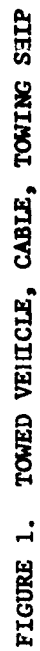
Other items which are taken to be known are the length, mass and diameter of the circular cable, and the mass and dimensions of the vehicle.

APPROXIMATE PHYSICAL SYSTEM

The cable is assumed to be inextensible (does not stretch) but its shape is flexible in form.

The length of the cable is divided into v integral number of segments as shown in Figure 1. The sections are imagined to be short rigid cylinders with universal joints at the junctions. The segments are numbered according to the subscript λ . $\lambda = 1$ for the segment next to the towing platform and $\lambda = v + 1$ is the underwater vehicle. The assumption of discrete divisions is a reasonable one, and its validity improves as v increases. The difference between the performance of the actual and assumed cable is expected to be small as v becomes large. A precedent for this assumption appears in Reference 1.

1. The first part of the paper discusses the importance of the research and the objectives of the study. It highlights the need for a comprehensive understanding of the research topic and the role of the research in advancing knowledge in the field.



In the problem considered the cable and vehicle do not roll. This will occur if either

(1) the vehicle is roll stabilized, or

(2) the shape of the bail on the vehicle is an arc of a circle the center of which is on the longitudinal axis of the vehicle, and the cable attachment is allowed to swivel so there is no torque transmitted from the laid cable to the vehicle.

If either of these conditions is satisfied then roll equal zero is a solution of the equations.

The vectors in the following problem have one of two distinct notations: (1) the customary vector bar, or (2) a subscript indicating its components. The first method will be used when possible.

The validity of a vector equation is independent of the particular coordinates in which it is expressed, so that coordinates are generally left unspecified. If the vehicle moves through an undisturbed fluid medium, the fixed frame may be taken relative to the earth. However, if the fluid medium moves with a velocity, it is convenient to fix the reference frame to an undisturbed particle moving with the medium.

Let \bar{x}_λ be the radius vector from the origin of such a fixed frame to the forward terminal point of the λ th element. To describe the motion of the system, it will be necessary to specify the motion of each cable element and also the motion of the underwater vehicle. The velocity and acceleration of each of the terminal points relative to the inertial frame are $\dot{\bar{x}}_\lambda(t)$ and $\ddot{\bar{x}}_\lambda(t)$. The position of the bow of the underwater vehicle relative to the ship is denoted by $\bar{x}_{v/s}$, where $\bar{x}_{v/s} = \bar{x}_{v+1} - \bar{x}_1$. It is not easy to define a useful quantity "diversion" while the towing ship is maneuvering. However if the towing ship is not maneuvering, the lateral diversion or distance can be shown to be:

$$|\bar{s}_0 \times \{\bar{x}_{v/s} - (\bar{k} \cdot \bar{x}_{v/s})\bar{k}\}|$$

where \bar{s}_0 is a unit vector pointing from the bow of the towing ship toward the stern, and \bar{k} is a unit vector along the x_3 axis in the inertial frame, as shown in Figure 1. The rate of diversion of vehicle relative to ship is then

$$d |\bar{s}_0 \times \{\bar{x}_{v/s} - (\bar{k} \cdot \bar{x}_{v/s})\bar{k}\}| / dt$$

Further, the quantity $\bar{k} \cdot \bar{x}_{v+1}$ represents the position of bow of the vehicle relative to the sea bottom, providing the inertial frame is placed on the sea bottom. Likewise the quantity $\bar{k} \cdot \dot{\bar{x}}_{v+1}$ would represent the rate of vertical displacement of bow of vehicle relative to the above-mentioned frame. The tensions in the cable at the towing ship and at the towing vehicle are $T_1(t)$ and $T_{v+1}(t)$ respectively, and at the forward end of λ th element, the tension is $\bar{T}_\lambda(t)$. In a similar manner other quantities can be formulated. These quantities are obtained from a solution of differential equations of motion of the cable-vehicle system. The formulation of the equations of motion is given on Page 23. The effect of the accelerated surrounding fluid on both cable and vehicle is represented more accurately in the considerations originally due to Kirchhoff (Reference 3), and more simply by a three-dimensional generalization of the expression used by Walton and Polachek (Reference 1). The latter method is employed for cable elements, whereas for the vehicle two approaches are considered. The two approaches, designated as Approach A and Approach B, are described briefly as follows:

1. The force on the vehicle is calculated on the basis of the same assumptions as made for a force on a cable element. That is, the magnitude is proportional to the volume of fluid displaced, and if \bar{a}_n is the normal component of the linear acceleration, the magnitude of the force is proportional to $|\bar{a}_n|$, and its direction is $-\bar{a}_n$. The proportionality constant for the vehicle may have to be determined experimentally. This approach has the undesirable feature that translation and rotation are not coupled through the reaction of the accelerated fluid.

2. The force on the vehicle is given by hydrodynamical considerations originally due to Kirchhoff. Physically the force is due to the accelerated motion generating fluid pressure variations over the surface, hence translation and rotation are coupled by the motion of the fluid. This approach has the undesirable feature that it complicates a formulation already lengthy.

The equations of motion of the vehicle using either Approach A or Approach B are developed separately. The resulting systems of equations differ from each other considerably in form. The testing of the two hypotheses implies a large experimental program. However, both are within the framework of a reasonably accurate formulation of physical facts.

To determine the time varying quantities such as diversion, instantaneous cable tensions and path of vehicle, we begin with a discussion of the dynamical equations of motion of a rigid body. The first of the following two equations describes the translation of mass

center and its relationship to the resultant external force. The second equation describes the rotation of the body, and specifically it relates the resultant external moment to the change in angular momentum. F and M with suitable subscripts are the forces and moments respectively acting on a body, H is the angular momentum of a body and u is the velocity of the mass center of a rigid body. The two equations are:

$$m_{\lambda} (du_{\lambda i}/dt) = F_{\lambda i}, (\lambda = 1, 2, \dots, v+1; i = 1, 2, 3)$$

$$dH_{\lambda i}/dt = M_{\lambda i}, (\lambda = 1, 2, \dots, v+1; i = 1, 2, 3)$$

The subscript λ refers to the λ th segment of the cable-body system, and the index i refers to the component in the direction of the axes of the fixed frame of reference. For example, $F_{\lambda i}$ is the i th component of all forces exerted on this element. The reader unaccustomed to the use of subscripts to denote the component of a vector should refer to Appendix B. While both equations are defined relative to a space fixed frame $x_1 x_2 x_3$, the second equation can also be written with respect to axes fixed to the center of mass of any body, providing all quantities entering into it are referred to this point. The force $F_{\lambda i}$ may be expressed as follows:

$$F_{\lambda i} = F_{A\lambda i} + F_{V\lambda i} + F_{G\lambda i} + (FIN)_i,$$

where $F_{A\lambda i}$ is the sum of the hydrodynamic forces which depend in any way on the accelerations of the coordinates of the end points of the elements. $F_{V\lambda i}$ is, with the exception of $(FIN)_i$, the sum of all hydrodynamic forces which depend on the first time derivatives but not on the second time derivatives, $F_{G\lambda i}$ is the sum of the forces which depend only on the properties of the element and on its orientation, and $(FIN)_i$ is the force exerted on the vehicle due to the deflection of the fins and velocity of the water.

THE FORCE $F_{A\lambda i}$

An accelerated vehicle produces accelerations of the surrounding fluid and so changes its momentum. The rate of change of linear (or angular) momentum is proportional to the force (or moment) producing this change which in turn is proportional to the linear (or angular) acceleration of the vehicle. Thus the vehicle behaves as if its mass were increased. This phenomenon is termed "mass accession" or added masses and added mass moments of inertia.

In Reference 1, Walton and Polachek assume that a hydrodynamic force is proportional to that component of acceleration of the center of gravity of the cable segment which is normal to this segment. This

may be generalized to three dimensions, and has an obvious application to a long thin cylinder, i.e., a cable segment. However when we come to the angular momentum equation, there is no allowance made for the accompanying added mass moment of inertia. Approach A applies this analysis to the towed vehicle and the cable segments.

As formulated by Kirchhoff the forces are linear combinations of the accelerations. The coefficients of this linear form are surface integrals of the velocity potential which describe the fluid flow. However, some of the coefficients or "added masses" may be obtained experimentally. Approach B applies these considerations to the towed vehicle.

THE FORCE $F_{V\lambda 1}$

There are two hydrodynamic forces due to the velocity of a cable segment through the water. They are the forces normal and tangential to the cable. For each of these we generalize the expressions used by Pote (Reference 2) to three dimensions. The assumptions made here are as follows:

1. The tangential force has the direction of \bar{s}_λ , where \bar{s}_λ is discussed in Appendix A.
2. The magnitude of the tangential force is independent of the orientation of the cable segment.
3. The magnitude of tangential force is $(1/2)\rho \bar{u}_\lambda^2 \cdot (\text{constant})$
 $A = C_1 \bar{u}_\lambda^2$ where \bar{u}_λ is the velocity of the segment through the water, and A is the wetted area of the segment.
4. The normal force is in the plane of the concurrent vectors \bar{s}_λ and \bar{u}_λ , and is normal to \bar{s}_λ .
5. The magnitude of the normal force is the product of two factors $\chi(\bar{u}_\lambda) \psi(\varphi_\lambda)$, where φ_λ is the angle between \bar{s}_λ and \bar{u}_λ . Further $\psi(\varphi_\lambda) = \sin^2 \varphi_\lambda$, and $\chi(\bar{u}_\lambda) = (1/2) \rho \bar{u}_\lambda^2 A (\text{constant}) = C_2 \bar{u}_\lambda^2$. (Refer to Appendix D for a more detailed discussion.)

THE FORCE $F_{G\lambda 1}$

There are two effects that contribute to $F_{G\lambda 1}$. The first is the weight of the cable in water. The second is composed of the tensions

$\bar{T}_\lambda, -\bar{T}_{\lambda+1}$, at each end transmitted by adjacent segments. This we suppose to be propagated instantaneously. Since action is equal to reaction and since segments of finite length will have finite angles of intersection, this force will not be along the axis of the segment. A diagram of these forces with an explanatory notation is contained in Appendix C. The possibility of the vehicle exerting thrust, such as from a propeller, may be included by setting $-\bar{T}_{v+2} = -T_p^s \bar{v}_{v+1}$, where T_p is the magnitude of the propeller thrust.

THE FORCE (FIN)₁

The lift and "drag" components of the resultant hydrodynamic force on the fins are assumed to follow quasi-steady and real-flow formulations. Forces which are zero when the rate of rotation of the fins is zero are neglected. The fins on the vehicle are in both the vertical and horizontal planes.

Let \bar{v}_{vf} denote the velocity of the vertical fins through the water, and \bar{v}_{vfp} denote the projection of \bar{v}_{vf} on the $\xi_1 \xi_2$ plane. (Refer to Appendix F). The following assumptions are made:

1. The force is called lift if it is normal to the plane containing both \bar{v}_{vfp} and \bar{n} ; and a force is called drag if it is in the direction $-\bar{v}_{vfp}$. Lift is positive when it tends to increase the $x_{v+1,2}$ coordinate from its initial steady state value.
2. Both of the forces are proportional to $|\bar{v}_{vfp}|^2$.
3. Each force is a function of the angle of attack; where the lift is an odd function and the drag an even function of the angle of attack.
4. The factors of proportionality in these forces are C_3 and C_4 for lift and drag, respectively. Then C_3 and C_4 include variations due to fin area and variations in C_L and C_D , which are coefficients of lift and drag respectively.

The forces on the vehicle due to horizontal fins are considered in exactly the same way. The lift due to these fins is normal to \bar{v}_{hf} (the velocity through the water of the horizontal fins), and is in the vertical plane. The horizontal and vertical fins are taken to be at the same distance from the bow, although this assumption is not necessary

and may be dropped for different vehicle designs. However this assumption implies $\bar{v}_{hf} \approx \bar{v}_{vf}$. For this pair of fins we obtain two other constants, C_5 and C_6 , for the lift and drag, respectively. Also the lift is considered to be positive when it tends to increase the $x_{v+1,3}$ coordinate from its initial steady state value. A detailed treatment is found in Appendix F.

Now consider the second set of equations of motion which relate the changes of angular moments of the vehicle or of the cable elements to the external applied moments. These equations are:

$$\dot{H}_{\lambda i} = M_{\lambda i}$$

In all cases the angular momentum and moment of the forces are taken about the center of gravity of the segment involved. As shown in Appendix B,

$$\dot{H}_{\lambda i} = \dot{\omega}_{\lambda i} I_{jj}^{\lambda} - \dot{\omega}_{\lambda j} I_{ji}^{\lambda} - e_{ist} \omega_{\lambda j} \omega_{\lambda s} I_{jt}^{\lambda}$$

where $\omega_{\lambda j}$ is the angular velocity of the cable element and $I_{ij}^{\lambda} = \sum m \rho_i \rho_j$.

The quantity ρ_i is the i th component of the vector from the c.g. to a typical small mass m and the summation is carried out over the λ th element. The above equation is referred to a frame whose origin is attached to the center of gravity and whose axes are parallel to the axes of the inertial frame.

To apply the above equation to the vehicle, it is more convenient to refer the rate of change of moment of momentum to a set of axes imbedded in the vehicle. Let the superscript prime denote the quantities in this frame, then the above relation becomes:

$$\dot{H}'_{v+1,i} = \dot{\omega}'_{v+1,i} I'_{jj}{}^{v+1} - \dot{\omega}'_{v+1,j} I'_{ji}{}^{v+1} - e_{ist} \omega'_{v+1,j} \omega'_{v+1,s} I'_{jt}{}^{v+1}$$

To obtain ω' , I' and the H' relative to the body fixed frame, let the rotation of the vehicle relative to the fixed inertial frame be described by a transformation of coordinates as follows:

$$x_i = c_{ij} \xi_j \text{ or } \xi_k = c_{ik} x_i$$

The c_{ij} are obtained in terms of previously defined quantities as shown in Appendix G. Since

$$\dot{H}_{v+1,i} = c_{in} \dot{H}'_{v+1,n}$$

and

$$\dot{\omega}_{v+1,i} = c_{in} \dot{\omega}'_{v+1,n}, \quad \dot{\omega}'_{v+1,j} = \dot{\omega}_{v+1,\ell} c_{\ell j}$$

$$\dot{\omega}'_{v+1,j} = \dot{\omega}_{v+1,\ell} c_{\ell j}, \quad \dot{\omega}'_{v+1,s} = \dot{\omega}_{v+1,k} c_{ks}$$

the preceding expression for $\dot{H}_{v+1,i}$ relative to the body fixed frame now takes the following form:

$$\begin{aligned} \dot{H}_{v+1,i} = & \dot{\omega}_{v+1,i} I_{jj}^{v+1} - \dot{\omega}_{v+1,\ell} c_{in} c_{\ell j} I_{jn}^{v+1} \\ & - c_{nst} \ddot{\omega}_{v+1,\ell} \ddot{\omega}_{v+1,k} c_{in} c_{\ell j} c_{ks} I_{jt}^{v+1} \end{aligned}$$

The products and moments of inertia relative to the body fixed frame are treated in Appendix B.

The value of $\dot{H}_{\lambda i}$ for $\lambda = 1, 2, \dots, v$ (i.e., for cable segments) is more readily obtained, since the I_{ji} depend on $\bar{s}_{\lambda i}$ by a known functional relation. In fact they finally simplify to:

$$\dot{H}_{\lambda i} = (\omega_{\lambda}^2 / 12) \dot{\omega}_{\lambda i}$$

The right side of the angular momentum equations contains the applied external moments with respect to the c.g. of the body. For convenience we express these in a manner similar to that in which the forces were decomposed.

$$M_{\lambda} = M_{G\lambda i} + M_{V\lambda i} + M_{A\lambda i} + (FINM)_i$$

where:

$M_{G\lambda i}$ is the moment of those forces which depend on neither the velocities nor accelerations,

$M_{V\lambda i}$ is the hydrodynamic moment of those forces other than those due to the fins depending on the velocities but not on the acceleration,

$M_{A\lambda i}$ is the hydrodynamic moment of all forces dependent on acceleration,

$(FINM)_i$ occurs only for the vehicle and is the moment due to the fins.

THE MOMENT $M_{G\lambda i}$

For a cable element this moment is due only to forces exerted by adjacent elements. For the vehicle we recall that in some designs there may be a propeller thrust whose line of action does not pass through the c.g. of the vehicle. Should this be the case, the additional moment of the propeller thrust about the c.g. must be included. Furthermore there occurs an external moment on the vehicle whenever the center of gravity and center of buoyancy do not coincide. Appendix C may be consulted for details.

THE MOMENT $M_{V\lambda i}$

The physical assumptions relating the forces due to the velocity have already been listed. Since translation and rotation of a cable element occur simultaneously, one end travels through the water faster than the other. This motion creates an opposing hydrodynamic moment. The computation of this moment is carried out in Appendix D. The moment of forces due to velocity on the vehicle is treated in an entirely similar manner.

THE MOMENT $M_{A\lambda i}$

The hydrodynamic moment is zero according to the hypothesis stated in Reference 1. This holds for all elements designated by $\lambda = 1, 2, \dots, v$, and also for the vehicle designated by $\lambda = v + 1$ in Approach A. However for the formulation discussed in Approach B, there is a moment on the vehicle due to the acceleration of the medium. In this instance the moment is calculated in exactly the same manner as the previously discussed hydrodynamic force. The details are found in Appendix E.

THE MOMENT $(FINM)_1$

The moment acts only on the vehicle. The fins are symmetric airfoils. $(FINM)_1$ is the moment about the c.g. due to the lift and drag on the two pairs of fins. The exact expression is contained in Appendix F.

We now consider the mathematical approach to the problem.

THE MATHEMATICAL PROBLEM

The equations to be solved are:

$$m_{\lambda} (du_{\lambda i}/dt) = F_{\lambda i}, \quad (\lambda = 1, 2, \dots, v+1; i = 1, 2, 3)$$

$$dH_{\lambda i}/dt = M_{\lambda i}, \quad (\lambda = 1, 2, \dots, v+1; i = 1, 2, 3)$$

where $du_{\lambda i}/dt$ is the linear acceleration of c.m. of the λ th element, and $dH_{\lambda i}/dt$ is an expression linear in the angular accelerations of the λ th element. In connection with these equations there are two fundamental considerations.

1. The right side of any equation of the first set contains $T_{\lambda i} - T_{\lambda+1, i} + F_{\lambda i}^*$, and the right side of any equation of the second set, except for $\lambda = v+1$, may be written $-k_{ijk} s_{ij} [T_{\lambda k} + T_{\lambda+1, k}] + M_{\lambda i}^*$, where F_i^* and M_i^* are expressions which do not contain the terms $T_{\lambda i}$. By summing the first set of equations from $\lambda = v+1$ to $\lambda = \mu$, an expression for $T_{\mu i}$ is obtained. Thus $T_{\mu i} + T_{\mu+1, i}$ (or $T_{\lambda i} + T_{\lambda+1, i}$) may be obtained. When this is substituted in the equation for angular momentum all $T_{\lambda i}$ are eliminated, and a set of equations in k and X is obtained. This set can be integrated using numerical processes.

2. The second fundamental consideration is that there are constraints among the elements. Suppose we make the natural choice and take as coordinates the position of the ends of the elements. We automatically have a connected chain of elements. The constraints are that each element has a given constant length.

The position of a rigid body is completely determined by the location of any three of its points, provided that these points are noncolinear. The distances between these points do not change. Hence there are three constraints among the nine coordinates and six degrees of freedom.

In the problem under consideration suppose for the moment that the ends of the rigid element are not connected to the adjacent elements. Suppose the coordinates of the two end points of the element are specified. Because there is one constraint among the six coordinates, five degrees of freedom remain. When the location of either vehicle or cable element is specified by giving the positions of its end points, the degree of freedom of rotation about a line joining the two points is lost because roll is assumed to be zero. The condition of zero roll, $\bar{\omega}_{\lambda j} \cdot \bar{s}_{\lambda j} = 0$,

yields a relation among the three angular velocities referred to the initial system. Further, there will be a relation among the three components of angular acceleration. Then for any element any component of the angular velocity may be found in terms of the other two. This is also true for the angular acceleration, and therefore also true for the angular momentum and its time derivative.

When, as in the preceding, the roll about the longitudinal axis is zero, the angular velocity may be readily determined in terms of the positions of the end points of the element and their derivatives. The expression obtained for $\omega_{\lambda 1}$ is dependent upon the fact that the element has a constant length (c.f. Appendix A); i.e., $d(\text{length of element})/dt = 0$. Similarly the relationship of $\phi_{\lambda 1}$ to the coordinates of the end points and their derivatives is dependent upon the constant length of the element. Therefore there is a dependency relation among the three equations of angular momentum and the condition that the length of the element is constant.

The preceding discussion was under the hypothetical condition that the ends of the element were free, which implied five degrees of freedom. In our problem the ends of the element are not free. The forward end of the $\lambda + 1$ st element must coincide with the after end of the λ th element. This eliminates three degrees of freedom for the $\lambda + 1$ st element, so that each element has two degrees of freedom.

For example the element next to the towing vehicle has a given length, and the position of its forward end is specified. Clearly the position of its after end lies on a sphere centered about the forward end, and may be specified, for instance, by giving its latitude and longitude. When these two values are specified for the first element, two more must be given to determine the position of the second element, and so forth.

Thus there are two degrees of freedom for each element and three angular momentum equations, of which only two are independent when account is taken of the constancy of length of an element.

We may use these two angular momentum equations for the two independent angular velocities and obtain the third angular velocity from the constraint. On the other hand, any nonindependent equation may be dropped from a dependent system. To drop the constraint it is sufficient to know that the three remaining equations are not independent of the constraint. As mentioned previously and as shown in Appendix A, the expressions obtained for $\omega_{\lambda 1}$ and $\phi_{\lambda 1}$ are dependent on the constraint, so that instead of using two angular momentum equations and the constraint, we may use the three angular momentum equations.

DISCUSSION OF METHOD OF SOLUTION OF EQUATION OF MOTION

Having taken as coordinates the end points of the elements, the w 's and u 's which occur in the equations of motion are obtainable in terms of derivatives of these coordinates. The $3(v+1)$ linear momentum equations determine the $3(v+1)$ forces $T_{\lambda i}$, and the $3(v+1)$ angular momentum equations then determine the $3(v+1)$ coordinates of the end points of the $v + 1$ elements.

The final $3(v+1)$ equations turn out to be linear in the linear accelerations. At this point we anticipate using a subroutine for solving differential equations such as SHARE program GL AIDE 1 (Reference 4). This necessitates a sequence of availability of information which is obtained by using previous values of positions or velocities to obtain the present value of acceleration. Thus we use known values of positions and velocities to compute the coefficients of the linear equations for the accelerations and the constants on the right sides of these equations. The linear system may then be solved numerically to obtain values for the acceleration. This satisfies the requirements of the numerical program.

EQUATIONS OF MOTION FOR APPROACH A AND APPROACH B

In this part the expressions for forces and moments which act on the vehicle are substituted into the equations of motion. The final differential equations of motion for Approach A are denoted Equations (1) and (2), and those for Approach B are denoted by Equations (3) and (4). A solution of either set of equations would yield the components of velocity of the displacement coordinates at each of the cable joints and also at the bow of vehicle.

Both formulations are now presented. The first formulation considers the assumptions of Approach A, and the second formulation considers the assumptions of Approach B.

The Equations of Motion for Approach A

The equations of linear momentum are:

$$\text{for the cable: } \frac{w_{\lambda} d_{\lambda}}{g} \frac{\ddot{x}_{\lambda+1} + \ddot{x}_{\lambda}}{2} = \ddot{x}_{\lambda} - \ddot{x}_{\lambda+1} + (w_{\lambda} - b_{\lambda}) d_{\lambda} \ddot{x} + c_1 d_{\lambda} \ddot{u}_{\lambda} \ddot{x}_{\lambda} + c_2 d_{\lambda} (\ddot{a}_{\lambda} \ddot{x} - \ddot{u}_{\lambda})^2 \ddot{x}_{\lambda} - \ddot{a}_{\lambda} \left(\frac{\ddot{x}_{\lambda+1} + \ddot{x}_{\lambda}}{2} - \ddot{x}_{\lambda} \right) \cdot \frac{\ddot{x}_{\lambda+1} + \ddot{x}_{\lambda}}{2} \ddot{x}_{\lambda} \quad (\lambda = 1, 2, \dots, v)$$

$$\text{for the vehicle: } w_{v+1} \frac{(d_{v+1} + d_{v+2})}{g} \left[\frac{d_{v+1} - d_{v+2}}{d_{v+1}} \ddot{x}_{v+1} + \frac{d_{v+2}}{d_{v+1}} \ddot{x}_{v+2} \right] =$$

$$\ddot{x}_{v+1} - \ddot{x}_{v+2} + (w_{v+1} - b_{v+1}) d_{v+1} \ddot{x} + c_1 d_{v+1} \ddot{u}_{v+1} \ddot{x}_{v+1} + c_2 d_{v+1} d_{v+2} (\ddot{a}_{v+1} \ddot{x} - \ddot{u}_{v+1})^2 \ddot{x}_{v+1} - \ddot{a}_{v+1} \left(\frac{\ddot{x}_{v+2} + \ddot{x}_{v+1}}{2} - \ddot{x}_{v+1} \right) \cdot \frac{\ddot{x}_{v+2} + \ddot{x}_{v+1}}{2} \ddot{x}_{v+1} +$$

where

$$w_{v+1} = \frac{\text{weight of the vehicle}}{\text{length of the vehicle}}$$

Since $\ddot{a}_{\lambda} = 1$ we have $\ddot{a}_{\lambda} \cdot \ddot{x}_{\lambda} = 0$ and $\ddot{a}_{\lambda} \cdot \ddot{x}_{\lambda} = -\ddot{x}_{\lambda}^2$ for any λ . Thus $\ddot{a}_{\lambda} \cdot (\ddot{x}_{\lambda+1} - \ddot{x}_{\lambda}) = -d_{\lambda} \ddot{x}_{\lambda}^2$.

$$-\ddot{a}_{\lambda} \cdot \frac{\ddot{x}_{\lambda+1} + \ddot{x}_{\lambda}}{2} = \ddot{a}_{\lambda} \cdot \frac{\ddot{x}_{\lambda}}{2} + \left[\ddot{a}_{\lambda} \cdot \ddot{x}_{\lambda} - d_{\lambda} \ddot{x}_{\lambda}^2 \right] = -\ddot{x}_{\lambda} \cdot \ddot{x}_{\lambda} - \frac{d_{\lambda}}{2} \ddot{x}_{\lambda}^2$$

The quantity \ddot{C}_{λ} is defined by $\ddot{C}_{\lambda} \ddot{x}_{\lambda} (-\frac{d_{\lambda}}{2} \ddot{x}_{\lambda}^2) + (w_{\lambda} - b_{\lambda}) d_{\lambda} \ddot{x} + c_1 d_{\lambda} \ddot{u}_{\lambda} \ddot{x}_{\lambda} + c_2 d_{\lambda} (\ddot{a}_{\lambda} \ddot{x} - \ddot{u}_{\lambda})^2 \ddot{x}_{\lambda} + \overline{FIN}$ for $\lambda = 1, 2, \dots, v + 1$,

where \overline{FIN} is zero unless $\lambda = v + 1$.

When the accelerations are brought to the left side the linear momentum equations are

$$\text{for the cable: } \frac{v_{\lambda+1}}{g} \frac{d}{dt} \left(\frac{\ddot{x}_{\lambda+1}}{2} + \frac{\ddot{x}_{\lambda}}{2} \right) + \frac{e_{\lambda}}{2} \left[\ddot{x}_{\lambda+1} + \ddot{x}_{\lambda} \right] - e_{\lambda} \ddot{x}_{\lambda} \cdot \ddot{x}_{\lambda} = \ddot{x}_{\lambda} - \ddot{x}_{\lambda+1} + \ddot{G}_{\lambda}, \quad (\lambda = 1, 2, \dots, v)$$

$$\text{for the vehicle: } \frac{v_{v+1}}{g} \frac{d}{dt} \left(\frac{d_{v+1}}{g} + d_{v+5} \right) \left[\frac{d_{v+1}}{d_{v+1}} \frac{d}{dt} \frac{\ddot{x}_{v+2}}{2} + \frac{d_{v+1}}{d_{v+1}} \frac{d}{dt} \frac{\ddot{x}_{v+1}}{2} \right] + \frac{e_{v+1}}{2} \left[\ddot{x}_{v+2} + \ddot{x}_{v+1} \right] - e_{v+1} \ddot{x}_{v+1} \cdot \ddot{x}_{v+1} = \ddot{x}_{v+1} - \ddot{x}_{v+2} + \ddot{G}_{v+1}$$

The form of the angular momentum equations will eventually make it necessary to introduce subscript notation for vectors. This is done now.

$$\text{Also for brevity we set } \mathbf{e}_{\lambda}^* = \frac{1}{2} \left[\frac{d_{\lambda}}{g} + e_{\lambda} \right], \quad (\lambda = 1, 2, \dots, v)$$

$$\mathbf{e}_{v+1}^* = \left[\frac{v_{v+1}}{g} (d_{v+1} + d_{v+5}) \left(\frac{d_{v+1}}{d_{v+1}} \frac{d}{dt} \frac{\ddot{x}_{v+2}}{2} + \frac{e_{v+1}}{2} \right) \right],$$

$$\mathbf{e}_{v+2}^* = \left[\frac{v_{v+1}}{g} (d_{v+1} + d_{v+5}) \frac{d}{dt} \frac{\ddot{x}_{v+2}}{2} + \frac{e_{v+1}}{2} \right].$$

$$b_{\lambda i j} = e_{\lambda}^* a_{i j} a_{\lambda j}, \quad \lambda = 1, 2, \dots, v+1, \text{ so that } b_{\lambda i j} = b_{\lambda j i}$$

Using these abbreviations, subscript notation for vectors, and the summation convention, the linear momentum equations are

$$\text{for the cable: } \mathbf{e}_{\lambda}^* \ddot{x}_{\lambda+1, i} + \mathbf{e}_{\lambda}^* \ddot{x}_{\lambda, i} - b_{\lambda i j} \ddot{x}_{\lambda j} \ddot{x}_{\lambda i} - \ddot{x}_{\lambda+1, i} + \ddot{G}_{\lambda i} \quad (\lambda = 1, 2, \dots, v; i = 1, 2, 3),$$

$$\text{for the vehicle: } \mathbf{e}_{v+2}^* \ddot{x}_{v+2, i} + \mathbf{e}_{v+1}^* \ddot{x}_{v+1, i} - b_{v+1, i j} \ddot{x}_{v+1, j} = \ddot{x}_{v+1, i} - \ddot{x}_{v+2, i} + \ddot{G}_{v+1, i} \quad \text{for } i = 1, 2, 3$$

$x_{1,i}$, the location of the towing platform, is not an unknown in the problem.

The $3(v+1)$ unknowns are the components of the $v+1$ vectors $\ddot{x}_2, \ddot{x}_3, \dots, \ddot{x}_{v+2,i}$.

The vehicle equation gives $T_{v+1,i} = T_{v+2,i} - G_{v+1,i} + a_{v+2,i}^* \ddot{x}_{v+2,i} + a_{v+1,i}^* \ddot{x}_{v+1,i} - b_{v+1,i} \ddot{x}_{v+1,i}$ ($i = 1, 2, 3$)

While for the cable $T_{\lambda i} = T_{\lambda+1,i} - G_{\lambda i} + a_{\lambda+1,i}^* \ddot{x}_{\lambda+1,i} + \epsilon_i^* \ddot{x}_{\lambda i} - b_{\lambda i} \ddot{x}_{\lambda i}$ ($\lambda = 1, 2, \dots, v; i = 1, 2, 3$).

When the equations for the $v, v-1, v-2, \dots, \mu$ cable segments are added to the equation

for the vehicle, the intermediate $T_{\lambda i}$ cancel. This leaves

$$\begin{aligned} T_{\mu i} &= T_{v+2,i} - \sum_{p=\mu}^{v+1} G_{pi} + \left[a_{v+2,i}^* \ddot{x}_{v+2,i} + a_{v+1,i}^* \ddot{x}_{v+1,i} + a_{v,i}^* \ddot{x}_{v,i} + a_{v-1,i}^* \ddot{x}_{v-1,i} + \dots + a_{\mu+1,i}^* \ddot{x}_{\mu+1,i} + a_{\mu,i}^* \ddot{x}_{\mu,i} \right] - \sum_{p=\mu}^{v+1} b_{pi} \ddot{x}_{pi} \\ &= T_{v+2,i} - \sum_{p=\mu}^{v+1} G_{pi} - \sum_{\lambda=\mu}^{v+1} b_{\lambda i} \ddot{x}_{\lambda i} + a_{v+2,i}^* \ddot{x}_{v+2,i} + a_{v+1,i}^* \ddot{x}_{v+1,i} + \sum_{\lambda=\mu+1}^{v+1} (a_{\lambda,i}^* + a_{\lambda-1,i}^*) \ddot{x}_{\lambda i} \quad (\mu = 1, 2, \dots, v) \end{aligned}$$

We make the convention that any \sum_i is zero when $i > j$. Then the above expression for $T_{\mu i}$ holds also for $\mu = v+1$, as

is apparent by comparison with the expression for $T_{v+1,i}$. Add the expressions for $T_{\mu i}$ and $T_{\mu+1,i}$ we obtain:

$$T_{\mu+1,i} + T_{\mu i} = \left[2T_{v+2,i} - 2 \sum_{\lambda=\mu+1}^{v+1} G_{\lambda,i} - G_{\mu i} \right] + 2a_{v+2,i}^* \ddot{x}_{v+2,i} + 2 \sum_{\lambda=\mu+2}^{v+1} (a_{\lambda,i}^* + a_{\lambda-1,i}^*) \ddot{x}_{\lambda i} - 2 \sum_{\lambda=\mu+1}^{v+1} b_{\lambda i} \ddot{x}_{\lambda i} + (2a_{\mu+1,i}^* + a_{\mu,i}^*) \ddot{x}_{\mu+1,i} + a_{\mu,i}^* \ddot{x}_{\mu,i} - b_{\mu+1,i} \ddot{x}_{\mu+1,i} - b_{\mu i} \ddot{x}_{\mu i}$$

$$= 2a_{v+2}^* \ddot{x}_{v+2,i} + 2 \sum_{\lambda=v+1}^{v+1} (\ddot{a}_{\lambda}^* + \ddot{a}_{\lambda-1}^*) \ddot{x}_{\lambda i} - 2 \sum_{\lambda=v+1}^{v+1} b_{\lambda i q} \ddot{x}_{\lambda q} - \ddot{a}_{\mu}^* (\ddot{x}_{\mu+1,i} - \ddot{x}_{\mu i}) - b_{\mu i q} \ddot{x}_{\mu q} + [2T_{v+2,i} - 2 \sum_{\lambda=v+1}^{v+1} G_{\lambda i} - G_{\mu i}] \quad (i = 1, 2, 3)$$

The derivatives with respect to the time of the angular momentum for a cable segment and for the vehicle are

$$\dot{H}_{\lambda i} = \dot{w}_{\lambda i} I_{jj}^{\lambda} - w_{\lambda j} I_{ji}^{\lambda} - e_{ist} w_{\lambda j} w_{\lambda s} I_{jt}^{\lambda} \quad (\text{for a cable segment}),$$

and

$$\dot{H}_{v+1,i} = \dot{w}_{v+1,i} I_{jj}^{v+1} - w_{v+1,i} I_{jj}^{v+1} - e_{nst} w_{v+1,i} w_{v+1,k} I_{jk}^{v+1} \quad \text{for the vehicle.}$$

$$\text{For a cable segment } I_{ij}^{\lambda} = \frac{m_{\lambda} d^2}{12} a_{\lambda i} a_{\lambda j} \text{ and } e_{ist} w_{\lambda j} w_{\lambda s} \left(\frac{m_{\lambda} d^2}{12} \right) a_{\lambda j} a_{\lambda t} = w_{\lambda j} a_{\lambda t} e_{ist} a_{\lambda s} \lambda_{ct}, \text{ which is zero since } w_{\lambda j} a_{\lambda j} = 0.$$

Also using $\dot{w}_{\lambda j} a_{\lambda j} = e_{jpq} a_{\lambda p} a_{\lambda q} \lambda_j = 0$ which is due to the anti-symmetry of e_{jpq} , we obtain for a cable element

$$\dot{H}_{\lambda i} = \frac{m_{\lambda} d^2}{12} a_{\lambda j} a_{\lambda j} = \frac{m_{\lambda} d^2}{12} \dot{a}_{\lambda i}^2$$

For the vehicle $\dot{H}_{v+1,i} = \dot{w}_{v+1,i} I_{jj}^{v+1} - w_{v+1,i} I_{jj}^{v+1} - e_{nst} w_{v+1,i} w_{v+1,s} I_{js}^{v+1} \cdot \dot{a}_{ct}^{v+1}$ where I_{ij}^{v+1} are evaluated in the body fixed system.

The following abbreviations will simplify the equations of angular momentum.

$$M_{\lambda i} = C_2 \frac{d^3}{6} [\bar{u}_\lambda \cdot \bar{\omega}_\lambda \times \bar{s}_\lambda] e_{ijk} a_{\lambda j} q_{\lambda k} \quad (\lambda = 1, 2, \dots, n+1, \quad i = 1, 2, 3)$$

which is the i th component of the moment on the λ th element due to the velocity of the water.

$$M_{Bi} = e_{ijk} a_{n+1,j} (d_{n+6} - d_{n+2}) b_{n+1} d_{n+1} \delta_{jk} \quad (i = 1, 2, 3),$$

which is the moment on the vehicle due to the displacements of center of buoyancy and center of gravity.

Then the equations of angular momentum are

$$\text{for a cable element: } \frac{\gamma d^3}{12g} \dot{\omega}_{\lambda i} = -\frac{1}{2} d_{\lambda} e_{ijk} a_{\lambda j} (\tau_{\lambda k} + \tau_{\lambda+1,k}) + C_2 \frac{d^3}{6} [\bar{u}_\lambda \cdot \bar{\omega}_\lambda \times \bar{s}_\lambda] e_{ijk} a_{\lambda j} q_{\lambda k}$$

$$= -\frac{1}{2} d_{\lambda} e_{ijk} a_{\lambda j} (\tau_{\lambda k} + \tau_{\lambda+1,k}) + M_{\lambda i} \quad (\lambda = 1, 2, \dots, n; \quad i = 1, 2, 3),$$

$$\text{and for the vehicle: } \dot{\omega}_{n+1,i} = \dot{\omega}_{n+1,i}^{(n+1)} - e_{ijn} c_{jn} \dot{\omega}_{n+1,j}^{(n+1)} - e_{nkn} c_{nk} \dot{\omega}_{n+1,k}^{(n+1)} =$$

$$e_{ijk} a_{n+1,j} [(d_{n+2} - d_{n+1}) \tau_{n+2,k} - d_{n+2} \tau_{n+1,k}] - e_{ijk} a_{n+1,j} (d_{n+6} - d_{n+2}) b_{n+1} d_{n+1} \delta_{jk} - C_2 \frac{d^3}{6} [\bar{u}_{n+1} \cdot \bar{\omega}_{n+1} \times \bar{s}_{n+1}] e_{ijk} a_{n+1,j} q_{n+1,k} =$$

$$e_{ijk} a_{n+1,j} [(d_{n+2} - d_{n+1}) \tau_{n+2,k} - d_{n+2} \tau_{n+1,k}] - M_{Bi} - M_{\lambda i}$$

The angular momentum equations are to be written with $T_{\lambda k} = T_{\lambda+1, k}$ eliminated, all terms containing accelerations on the left side, and all other terms on the right side. Thus,

$$\text{for the cable: } \frac{\gamma d^3}{12g} e_{ijk} s_{\lambda j} \frac{\ddot{x}_{\lambda+1, k} - \ddot{x}_{\lambda k}}{d_{\lambda}} + \frac{1}{2} d_{\lambda} e_{ijk} s_{\lambda j} \left[2\ddot{a}_{\lambda+2} \ddot{x}_{\lambda+2, k} + 2 \sum_{\mu=\lambda+1}^{\nu+1} (\ddot{a}_{\mu}^* + \ddot{a}_{\mu-1}^*) : \ddot{x}_{\mu k} - 2 \sum_{\mu=\lambda+1}^{\nu+1} b_{\mu k q} \ddot{x}_{\mu q} - \ddot{a}_{\lambda}^* (\ddot{x}_{\lambda+1, k} - \ddot{x}_{\lambda k}^*) - b_{\lambda k q} \ddot{x}_{\lambda q}^* \right]$$

$$= -\frac{1}{2} d_{\lambda} e_{ijk} s_{\lambda j} \left[2T_{\nu+2, k} - 2 \sum_{\mu=\lambda+1}^{\nu+1} G_{\mu k} - G_{\lambda k} \right] + VM_{\lambda i} \quad (\lambda = 1, 2, \dots, \nu; i = 1, 2, 3)$$

for the vehicle:

$$\begin{aligned} e_{ijk} s_{\nu+1, j} \ddot{s}_{\nu+1, k} &= e_{jmk} s_{\nu+1, m} s_{\nu+1, k} c_{in} c_{j\ell} I_{\ell n}^{\nu+1} - e_{nkt} s_{\nu+1, j} s_{\nu+1, t} e_{in}^c j^c s_{\ell k}^c I_{\ell}^{\nu+1} + e_{ijk} s_{\nu+1, j} d_{\nu+2} \left[\ddot{a}_{\nu+2} \ddot{x}_{\nu+2, k} + \ddot{a}_{\nu+1}^* \ddot{x}_{\nu+1, k} - b_{\nu+1, kq} \ddot{x}_{\nu+1, q}^* \right] \\ &= e_{ijk} s_{\nu+1, j} (d_{\nu+2} - d_{\nu+1}) T_{\nu+2, k} - EM_{\ell} - VM_{\nu+1, i} - e_{ijk} s_{\nu+1, j} d_{\nu+2} (T_{\nu+2, k} - G_{\nu+1, k}) \\ &= -e_{ijk} s_{\nu+1, j} \left[d_{\nu+1} T_{\nu+2, k} - d_{\nu+2} G_{\nu+1, k} \right] - EM_{\ell} - VM_{\nu+1, i} \quad (i = 1, 2, 3). \end{aligned}$$

This is a system of $3(\nu+1)$ equations for the $3(\nu+1)$ variables $\ddot{x}_{2, 1}, \ddot{x}_{2, 2}, \ddot{x}_{2, j}, \ddot{x}_{3, 1}, \dots, \ddot{x}_{\nu+2, 3}$

When written in terms of these variables, these equations read:

$$e_{ijk} d_{\lambda} s_{\lambda j} \left[\ddot{a}_{\nu+2} \ddot{x}_{\nu+2, k} + \left(\frac{\rho d_{\lambda}}{12g} - \frac{\ddot{a}_{\lambda}}{2} \right) \ddot{x}_{\lambda+1, k} + \sum_{\mu=\lambda+1}^{\nu+1} (\ddot{a}_{\mu}^* + \ddot{a}_{\mu-1}^*) \ddot{x}_{\mu k} - \sum_{\mu=\lambda+1}^{\nu+1} b_{\mu k q} \ddot{x}_{\mu q}^* - \left(\frac{\rho d_{\lambda}}{12g} - \frac{\ddot{a}_{\lambda}}{2} \right) \ddot{x}_{\lambda k} - \frac{b_{\lambda k q}}{2} \ddot{x}_{\lambda q}^* \right] =$$

$$-e_{ijk} d_{\lambda} s_{\lambda j} \left[T_{\nu+2, k} - \sum_{\mu=\lambda+1}^{\nu+1} G_{\mu k} - \frac{1}{2} G_{\lambda k} \right] + VM_{\lambda i} \quad (\lambda = 1, 2, \dots, \nu)$$

$$\left[e_{ijk} a_{v+1,j} \left(\frac{I^{v+1}}{d_{v+1}} + a_{v+2}^* d_{v+2} \right) - e_{jmk}^a v_{v+1,m} c_{in}^c c_{jl} \frac{I^{v+1}}{d_{v+1}} \right] \ddot{x}_{v+2,k} - \left[e_{ijk}^a v_{v+1,j} \left(\frac{I^{v+1}}{d_{v+1}} - a_{v+2}^* d_{v+2} \right) - e_{jmk}^a v_{v+1,m} c_{in}^c c_{jl} \frac{I^{v+1}}{d_{v+1}} \right] \ddot{x}_{v+1,k} - e_{ijk}^a v_{v+1,j} d_{v+2}^b v_{v+1,kq} \ddot{x}_{v+1,q}$$

Equation (2)

$$= -e_{nkc}^w v_{v+1,j}^w v_{v+1,s} c_{in}^c c_{jl}^c c_{sk}^c \frac{I^{v+1}}{d_{v+1}} - e_{ijk} \left[d_{v+1} \ddot{x}_{v+2,k} - d_{v+2} c_{v+1,k} \right] - M_{B1} - M_V, v_{v+1,i}$$

The Equations of Motion for Approach 2

The equations of linear momentum are:

$$\text{for the cable: } m_{\lambda} \ddot{x}_{\lambda+1,i} + \frac{\ddot{x}_{\lambda+1,i}}{2} = T_{\lambda i} - T_{\lambda+1,i} + B_{\lambda i} + FD_{\lambda i} + F_{\lambda i} + FA_{\lambda i} \quad (\lambda = 1, 2, \dots, v; i = 1, 2, 3),$$

$$\text{for the vehicle: } m_{v+1} \ddot{x}_{Gi} = T_{v+1,i} - T_{v+2,i} + B_{v+1,i} + FD_{v+1,i} + FL_{v+1,i} + FIM_1 + FA_{v+1,i} \quad (i = 1, 2, 3).$$

$$\text{where } x_{Gi} = \frac{d_{v+1} - d_{v+2}}{d_{v+1}} x_{v+1,i} - \frac{d_{v+2}}{d_{v+1}} x_{v+2,i}$$

and $B_{v+1,i}$ is the buoyant force on the vehicle.

$$\text{Define } Q_{\lambda i} = B_{\lambda i} + F_{\lambda i} + FL_{\lambda i}$$

$$Q_{v+1,i} = B_{v+1,i} + FD_{v+1,i} + FL_{v+1,i} + FIM_1$$

Add the equation for the vehicle to the equations for the λ^{th} cable segments for $\lambda = \mu, \mu + 1, \dots, v$. Then

$$T_{\mu i} - T_{v+2,i} = - \sum_{\lambda=\mu}^{v+1} Q_{\lambda i} - \sum_{\lambda=\mu}^{v+1} FA_{\lambda i} + \sum_{\lambda=\mu}^{v+1} m_{\lambda} \frac{\ddot{x}_{\lambda+1,i}}{2} + \sum_{\lambda=\mu}^{v+1} m_{\lambda+1} \ddot{x}_{Gi} \quad (\mu = 1, 2, \dots, v).$$

Add this equation to itself for two consecutive values of μ .

$$T_{\mu i} + T_{\mu+1,i} = 2T_{v+2,i} - 2 \sum_{\lambda=\mu+1}^{v+1} Q_{\lambda i} - Q_{\mu i} + 2 \sum_{\lambda=\mu+1}^{v+1} m_{\lambda} \frac{\ddot{x}_{\lambda+1,i}}{2} + 2 \sum_{\lambda=\mu+1}^{v+1} m_{\lambda+1} \ddot{x}_{Gi} + m_{\mu} \frac{\ddot{x}_{\mu+1,i}}{2} - 2 \sum_{\lambda=\mu+1}^{v+1} FA_{\lambda i} - F_{\mu i} \quad (1 \leq \mu \leq v-1).$$

The third and fourth terms on the right side equal $\sum_{\lambda=\mu+1}^{\nu} (m_{\lambda-1} + m_{\lambda}) \ddot{x}_{\lambda i} + m_{\nu} \ddot{x}_{\nu+1, i} - m_{\mu} \ddot{x}_{\mu-1, i}$ since $\sum_{\lambda=\mu+1}^{\nu} m_{\lambda} \ddot{x}_{\lambda+1, i} =$

$$\sum_{\lambda=\mu+1}^{\nu+1} m_{\lambda-1} \ddot{x}_{\lambda i} = \sum_{\lambda=\mu+1}^{\nu} m_{\lambda-1} \ddot{x}_{\lambda i} + m_{\nu} \ddot{x}_{\nu+1, i} - m_{\mu} \ddot{x}_{\mu+1, i}.$$

which contain terms of the form $\sum_{\lambda=\mu+1}^{\mu} \ddot{x}_{\lambda i}$ or $\sum_{\lambda=\mu+1}^{\nu} \ddot{x}_{\lambda i}$ when $\mu = \nu$. As in approach A we make the convention \sum_i^j is zero for $i > j$. Then

this expression $T_{\mu i} + T_{\mu+1, i}$ holds for $\mu = 1, 2, \dots, \nu$.

For the cable $I_{jj}^{\lambda} = \frac{m_{\lambda} d^2}{12} s_{\lambda j} s_{\lambda j} = \frac{m_{\lambda} d^2}{12} s_{\lambda i} s_{\lambda j} \dot{\omega}_{\lambda j}$. However $s_{\lambda j} \dot{\omega}_{\lambda j} = s_{\lambda j} e_{j k} s_{\lambda k} = s_{\lambda k} e_{k j i} s_{\lambda i} s_{\lambda j} = 0$, so that $\dot{\omega}_{\lambda j} I_{jj}^{\lambda}$

Also $e_{i a t} \omega_{\lambda j} I_{j t}^{\lambda} = e_{i a t} \omega_{\lambda s} \frac{m_{\lambda} d^2}{12} s_{\lambda t} s_{\lambda j} \cdot \omega_{\lambda j} = 0$ since $s_{\lambda j} \omega_{\lambda j} = 0$. The superscripts on the $I_{j t}^{\lambda}$ now have no further use and will be dropped.

Henceafter $I_{j i}$ and $I_{j i}^{\lambda}$ refer to the vehicle, and $I_{j i}^{\lambda}$ are measured in a frame fixed to the vehicle.

The angular momentum equations are:

$$\text{for the cable: } \frac{m_{\lambda} d^2}{12} \dot{\omega}_{\lambda i} = -e_{i j k} \frac{s_{\lambda j}}{2} [T_{\lambda k} + T_{\lambda+1, k}] + V M_{\lambda i} + A M_{\lambda i} \quad (\lambda = 1, 2, \dots, \nu; i = 1, 2, 3)$$

for the vehicle:

$$\dot{\omega}_{\nu+1, i} I_{j j}^{\nu+1} - e_{i n k} \dot{\omega}_{\nu+1, k} I_{j n}^{\nu+1} - e_{n s t} i n k j C_{m s} \omega_{\nu+1, k} \omega_{\nu+1, s} I_{j t}^{\nu+1} = e_{i j k} s_{\nu+1, j} [(d_{\nu+2} - d_{\nu+1}) \dot{\omega}_{\nu+2, k} - d_{\nu+2} \dot{\omega}_{\nu+1, k}] + B M_{\nu i} + V M_{\nu+1, i} + F I M_{\nu i} + A M_{\nu+1, i} \quad (i = 1, 2, 3)$$

In the above equations, $\ddot{\chi}_{\lambda i} = e_{ijk} a_{kj} \frac{\ddot{\chi}_{\lambda+1,k} - \ddot{\chi}_{\lambda k}}{d_{\lambda}}$;

$EM_{\lambda i}$, $VM_{\lambda+1,i}$, $FIM_{\lambda i}$ are moments about the center of gravity of the vehicle due to the buoyancy of the vehicle, the velocity of the water except as it affects the fins;

$AM_{\lambda i}$ is the moment on the vehicle due to the acceleration of the water.

For the vehicle,

$$FA_{\lambda+1,i} = -\rho c_{ij} c_{rk} C1_{jk} (\ddot{x}_{Gr} - e_{rpq} \ddot{w}_{\lambda+1,p} \ddot{x}_{Gq}) - \rho c_{ij} c_{rk} C3_{jk} \ddot{w}_{\lambda+1,p} \ddot{x}_{Gq} - 2e_{rpq} \ddot{w}_{\lambda+1,p} \ddot{x}_{Gq}$$

$$AM_{\lambda+1,i} = -\rho c_{ij} c_{rk} C3_{jk} (\ddot{x}_{Gr} - e_{rpq} \ddot{w}_{\lambda+1,p} \ddot{x}_{Gq}) - \rho c_{ij} c_{rk} C2_{jk} \ddot{w}_{\lambda+1,p} \ddot{x}_{Gq} - 2e_{rpq} \ddot{w}_{\lambda+1,p} \ddot{x}_{Gq}$$

For the cable,

$$FA_{\lambda i} = -\gamma_{\lambda} \left(\frac{\ddot{\chi}_{\lambda+1,i} - \ddot{\chi}_{\lambda i}}{2} - e_{\lambda j} \frac{\ddot{\chi}_{\lambda+1,i} + \ddot{\chi}_{\lambda i}}{2} e_{\lambda i} \right) \quad (\lambda = 1, 2 \dots n; i = 1, 2, 3);$$

$$AM_{\lambda i} = 0;$$

Next substitute in the angular momentum equations these expressions and the expression for $[\ddot{\chi}_{\lambda i} + \ddot{\chi}_{\lambda+1,i}]$ obtained previously. In this process all terms containing accelerations are written on the left side, and all other terms are written on the right side. The result is:

for the cable:

$$\frac{m_d^2}{12} e_{ijk} s_{ij} \frac{\ddot{x}_{i+1,k} - \ddot{x}_{jk}}{d_{ij}} + e_{ijk} \frac{s_{ij}}{2} \left[\sum_{\mu=\lambda+1}^v (\ddot{m}_{\mu-1} + \ddot{m}_{\mu}) \ddot{x}_{\mu k} + \ddot{m}_{\mu} \ddot{x}_{\mu+1,k} + 2\ddot{m}_{\mu+1} \ddot{x}_{\mu+1,k} - \frac{\ddot{m}_{\mu+1,k}}{2} + \frac{\ddot{m}_{\mu+2,k}}{2} - 2 \sum_{\mu=\lambda+1}^{v+1} -e_{\mu} \left(\frac{\ddot{x}_{\mu+1,k} + \ddot{x}_{\mu k}}{2} - \ddot{s}_{\mu k} - e_{\mu j} \frac{\ddot{x}_{\mu+1,j} + \ddot{x}_{jk}}{2} s_{\mu k} \right) \right]$$

$$-e_{\lambda} \left\{ \frac{\ddot{x}_{\lambda+1,k} + \ddot{x}_{\lambda k}}{2} - s_{\lambda j} \frac{\ddot{x}_{\lambda+1,j} + \ddot{x}_{jk}}{2} s_{\lambda k} \right\} = -e_{ijk} \frac{s_{ij}}{2} [2T_{v+2,k} - 2 \sum_{\mu=\lambda+1}^{v+1} \ddot{q}_{\mu k} - \ddot{q}_{\mu k}] \quad (\lambda = 1, 2, \dots, v; i = 1, 2, 3). \quad (\text{Equation 3})$$

for the vehicle:

$$e_{ijk} s_{v+1,j} \frac{\ddot{x}_{v+2,k} - \ddot{x}_{v+1,k}}{d_{ij}} - e_{kmn} s_{v+1,k} \frac{\ddot{x}_{v+2,l} - \ddot{x}_{v+1,l}}{d_{v+1}} c_{in k j}^c I_i^j + e_{ijk} s_{v+1,j} d_{v+2} \left[\ddot{m}_{v+1} \ddot{x}_{Gk} - \left\{ -\rho c_{km}^c c_{rn}^c \left(\ddot{x}_{Gr} - e_{rpq} \ddot{x}_{Gq}^e p_{uv} s_{v+1,u} \frac{\ddot{x}_{v+2,v} - \ddot{x}_{v+1,v}}{d_{v+1}} \right) \right. \right.$$

$$\left. - \rho c_{km}^c c_{ln}^c e_{mn} s_{vpq} s_{v+1,p} \frac{\ddot{x}_{v+2,q} - \ddot{x}_{v+1,q}}{d_{v+1}} \right] + \rho c_{ij}^c c_{rk}^c C_3 \left(\ddot{x}_{Gr} - e_{rpq} \ddot{x}_{Gq}^e p_{uv} s_{v+1,u} \frac{\ddot{x}_{v+2,v} - \ddot{x}_{v+1,v}}{d_{v+1}} \right) + \rho c_{ij}^c c_{jk}^c C_2 e_{jk} s_{pq} s_{v+1,p} \frac{\ddot{x}_{v+2,q} - \ddot{x}_{v+1,q}}{d_{v+1}}$$

$$= e_{nst}^c c_{in k j}^c m s_{v+1,k}^w s_{v+1,m}^w I_j^j + e_{ijk} s_{v+1,j} (d_{v+2} - d_{v+1}) T_{v+2,k} - e_{ijk} s_{v+1,j} d_{v+2} [T_{v+2,k} - \ddot{q}_{v+1,k} - \int -\rho c_{km}^c c_{rn}^c (e_{jpk} s_{qr} s_{v+1,p} s_{v+1,q} - 2e_{rpq} s_{v+1,p} s_{v+1,q}^w - 2e_{rpq} s_{v+1,p} s_{v+1,q}^w - 2e_{rpq} s_{v+1,p} s_{v+1,q}^w - 2e_{rpq} s_{v+1,p} s_{v+1,q}^w)]$$

$$+ B \ddot{M}_i + V \ddot{M}_{v+1,i} + F \ddot{M}_i - \rho c_{ij}^c c_{rk}^c C_3 (e_{jpk} s_{qr} s_{v+1,p} s_{v+1,q} - 2e_{rpq} s_{v+1,p} s_{v+1,q}^w) \quad \text{for } (i = 1, 2, 3). \quad (\text{Equation 4})$$

The preceding system contains $3v + 3$ equations. The $3(v+1)$ unknowns are $\ddot{x}_{2,i}, \ddot{x}_{3,i}, \dots, \ddot{x}_{v+2,i}$. The system is linear in the accelerations and

for a given t has constant coefficients, and therefore may be solved for the accelerations.

CONCLUSIONS

The equations of motion for a towed underwater vehicle have been derived. The cable was imagined to consist of a finite number of rigid cylinders. Two hypotheses were made relating the hydrodynamic force and moment on the vehicle and its acceleration, and the equations were developed separately.

The equations of motion are involved, the most desirable of the two hypotheses above being the most intricate. In these equations there occurs a sevenfold sum. Complexity of the equations is the factor which will limit the computability.

The principal restriction in the equations of motion is that the vehicle does not roll, nor does the cable. On the other hand, the equations are general in that both the towing and towed vehicle may move in a highly arbitrary fashion.

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APPENDIX A

GEOMETRY

This appendix contains, with the exception of the coordinate transformation in Appendix G, all of the diverse kinematical considerations relevant to the problem.

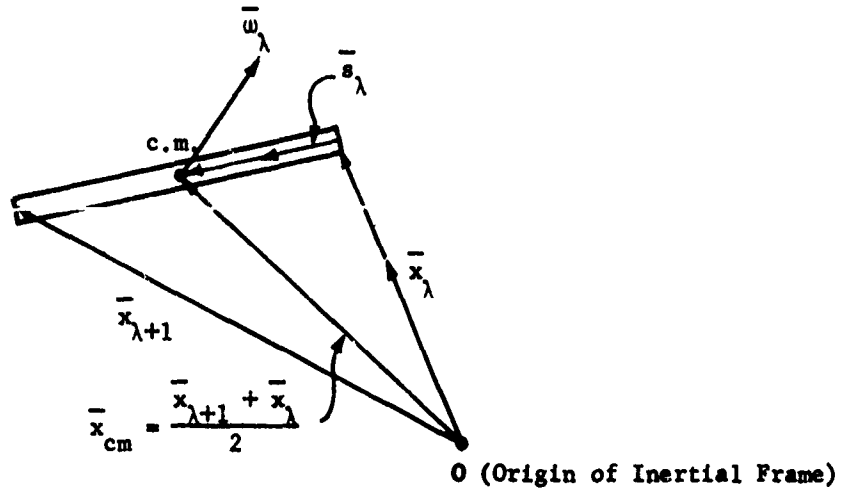


FIGURE A1. THE λ th CABLE SEGMENT

Let \bar{s}_λ be a unit vector tangent to the cable segment of length d_λ . Referring to Figure A1, and with the rules of vector addition, we write

$$\bar{x}_{\lambda+1} = \bar{x}_\lambda + d_\lambda \bar{s}_\lambda, \text{ or } \bar{s}_\lambda = (\bar{x}_{\lambda+1} - \bar{x}_\lambda)/d_\lambda$$

Since the length of cable is constant, then $d(1/d_\lambda)/dt = 0$, and the derivative of \bar{s}_λ with respect to time gives:

$$\dot{\bar{s}}_\lambda = (\dot{\bar{x}}_{\lambda+1} - \dot{\bar{x}}_\lambda)/d_\lambda \quad (A-1)$$

In a similar manner:

$$\dot{\bar{s}}_\lambda = (\dot{\bar{x}}_{\lambda+1} - \dot{\bar{x}}_\lambda)/d_\lambda$$

Since the center of mass and center of volume of a cable segment coincide, by the rules of kinematics, we obtain

$$\dot{\mathbf{x}}_{\lambda} = \dot{\mathbf{x}}_{cm} + \bar{\omega}_{\lambda} \times (d_{\lambda}/2) \bar{\mathbf{s}}_{\lambda}$$

$$\dot{\mathbf{x}}_{\lambda+1} = \dot{\mathbf{x}}_{cm} + \bar{\omega}_{\lambda} \times (-d_{\lambda}/2) \bar{\mathbf{s}}_{\lambda}$$

Subtracting these two equations we get

$$\dot{\mathbf{x}}_{\lambda} - \dot{\mathbf{x}}_{\lambda+1} = \bar{\omega}_{\lambda} \times d_{\lambda} \bar{\mathbf{s}}_{\lambda}$$

where $\bar{\omega}_{\lambda}$ is the angular velocity of the cable segment. The vector product with respect to $\bar{\mathbf{s}}_{\lambda}$ gives:

$$\bar{\mathbf{s}}_{\lambda} \times (\dot{\mathbf{x}}_{\lambda+1} - \dot{\mathbf{x}}_{\lambda}) = d_{\lambda} \bar{\mathbf{s}}_{\lambda} \times (\bar{\omega}_{\lambda} \times \bar{\mathbf{s}}_{\lambda}) = d_{\lambda} \bar{\omega}_{\lambda} - d_{\lambda} (\bar{\omega}_{\lambda} \cdot \bar{\mathbf{s}}_{\lambda}) \bar{\mathbf{s}}_{\lambda}$$

Since the cable segment is assumed not to roll, then $\bar{\omega}_{\lambda} \cdot \bar{\mathbf{s}}_{\lambda} = 0$.

Solving the previous equation for $\bar{\omega}_{\lambda}$, and with the aid of Equation A-1, we get $\bar{\omega}_{\lambda} = \bar{\mathbf{s}}_{\lambda} \times \dot{\bar{\mathbf{s}}}_{\lambda}$. Since $\bar{\mathbf{s}}_{\lambda} \times \dot{\bar{\mathbf{s}}}_{\lambda} = 0$, then it follows that $\dot{\bar{\mathbf{s}}}_{\lambda} = \bar{\omega}_{\lambda} \times \bar{\mathbf{s}}_{\lambda}$.

By referring to Figure A2, the location of the center of mass of vehicle, \mathbf{x}_G with respect to the inertial frame may be expressed in the following way:

$$\bar{\mathbf{x}}_G = (d_{v+1} - d_{v+2}) \bar{\mathbf{x}}_{v+1} / d_{v+1} + (d_{v+2} / d_{v+1}) \bar{\mathbf{x}}_{v+2}$$

Here the velocity and acceleration of the center of mass are the usual first and second time derivatives of \mathbf{x}_G .

The important unit vector $\bar{\mathbf{q}}_{\lambda}$, as shown in Figure A3, has the following properties: (a) it is in the plane of $\bar{\mathbf{u}}_{\lambda}$ and $\bar{\mathbf{s}}_{\lambda}$, (b) it is normal to $\bar{\mathbf{s}}_{\lambda}$, (c) it is in the sense that the direction between $\bar{\mathbf{q}}_{\lambda}$ and $-\bar{\mathbf{u}}_{\lambda}$ is less than $\pi/2$, and (d) it has unit length. The analytical statement of these conditions is:

$$1. \quad \bar{\mathbf{q}}_{\lambda} = \alpha \bar{\mathbf{u}}_{\lambda} + \beta \bar{\mathbf{s}}_{\lambda}, \text{ for some constants } \alpha \text{ and } \beta.$$

$$2. \quad \bar{\mathbf{q}}_{\lambda} \cdot \bar{\mathbf{s}}_{\lambda} = 0, \quad (3) \quad \bar{\mathbf{q}}_{\lambda} \cdot \bar{\mathbf{u}}_{\lambda} < 0, \quad (4) \quad \bar{\mathbf{q}}_{\lambda}^2 = 1$$

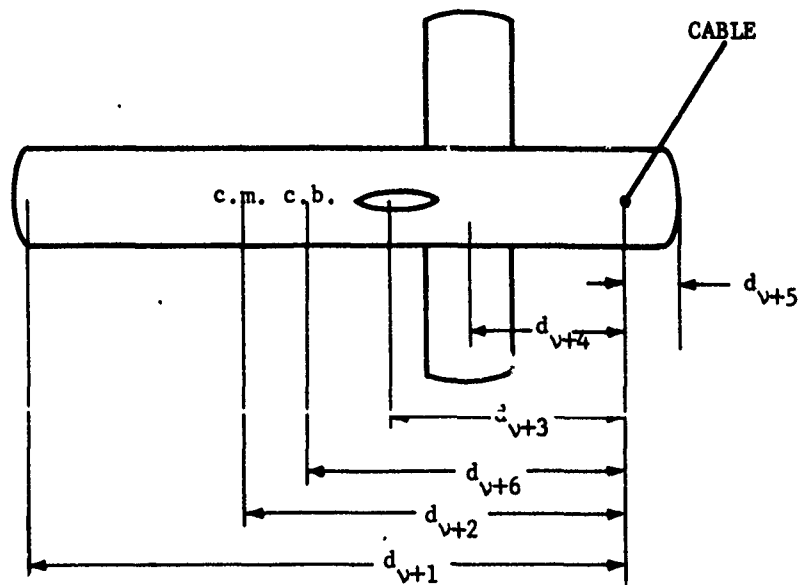


FIGURE A2. MEASUREMENTS ON VERTICAL PROJECTION OF VEHICLE

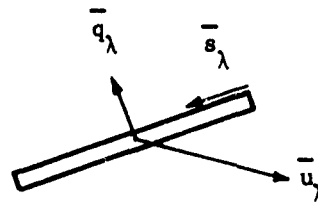


FIGURE A3. UNIT VECTORS \bar{q}_λ , \bar{s}_λ , AND VELOCITY \bar{u}_λ OF CABLE SEGMENT

When one takes the dot product of both sides of (1) with \bar{s}_λ , the left side is zero by (2); then

$$\alpha \bar{u}_\lambda \cdot \bar{s}_\lambda + \beta = 0$$

If this value of β is substituted in (1), the result is

$$\bar{q}_\lambda = \alpha[\bar{u}_\lambda - \bar{u}_\lambda \cdot \bar{s}_\lambda s_\lambda] = \alpha[\bar{s}_\lambda \times (\bar{u}_\lambda \times \bar{s}_\lambda)]$$

From condition (4) and with this expression for \bar{q}_λ , we obtain

$$\begin{aligned} 1 &= \alpha \bar{s}_\lambda \times (\bar{u}_\lambda \times \bar{s}_\lambda) \cdot \alpha \bar{s}_\lambda \times (\bar{u}_\lambda \times \bar{s}_\lambda) \\ &= \alpha^2 \bar{s}_\lambda \cdot (\bar{u}_\lambda \times \bar{s}_\lambda) \times [\bar{s}_\lambda \times (\bar{u}_\lambda \times \bar{s}_\lambda)] \\ &= \alpha^2 \bar{s}_\lambda \cdot [(\bar{u}_\lambda \times \bar{s}_\lambda) \cdot \bar{s}_\lambda] \end{aligned}$$

Since $\bar{s}_\lambda \cdot \bar{s}_\lambda = 1$, and solving for α , where γ is either ± 1 , we obtain

$$\alpha = \gamma / |\bar{u}_\lambda \times \bar{s}_\lambda|$$

To determine whether $\gamma = +1$ or $\gamma = -1$ is applicable, condition (3) may be used. Applying this condition, the following result is obtained:

$$\bar{q}_\lambda \cdot \bar{u}_\lambda = \gamma \bar{u}_\lambda \cdot \bar{s}_\lambda \times (\bar{u}_\lambda \times \bar{s}_\lambda) / |\bar{u}_\lambda \times \bar{s}_\lambda| < 0$$

Since the quantity after the symbol γ is positive, it follows that $\gamma = -1$. Therefore

$$\bar{q}_\lambda = -\bar{s}_\lambda \times (\bar{u}_\lambda \times \bar{s}_\lambda) / |\bar{u}_\lambda \times \bar{s}_\lambda|$$

Vehicle Geometry

- d_{v+1} is the distance from the cable attachment to the tail,
- d_{v+2} is the distance from the cable attachment to the c.g.,
- d_{v+3} is the distance from the cable attachment to the effective axis of the horizontal fins.
- d_{v+4} is the distance from the cable attachment to the effective axis of the vertical fins,
- d_{v+5} is the distance from the cable attachment to the nose ($d_{v+5} \geq 0$),
- d_{v+6} is the distance from the cable attachment to the center of buoyancy.

Let \bar{x} be the vector from the c.g. to the fin axis.

\bar{s}_{v+1} has the sense of a vector from the cable attachment to the c.g. If the d's are positive measured in the sense of \bar{s}_{v+1} , for any placement of the c.g., fins, and cable attachment, then:

$$\bar{x} = (d_{v+3} - d_{v+2}) \bar{s}_{v+1}, \quad (\text{horizontal fins})$$

$$\bar{x} = (d_{v+4} - d_{v+2}) \bar{s}_{v+1}, \quad (\text{vertical fins})$$

Summary of results

1. The linear velocity and acceleration of the center of λ th segment of cable with respect to the inertial frame are:

$$(\bar{x}_{\lambda+1} + \bar{x}_{\lambda})/2, \quad (\ddot{x}_{\lambda+1} + \ddot{x}_{\lambda})/2$$

2. The linear velocity and acceleration of the vehicle with respect to the inertial frame are:

$$\dot{\bar{x}}_G = a \dot{\bar{x}}_{v+1} + b \dot{\bar{x}}_{v+2}$$

$$\ddot{\bar{x}}_G = a \ddot{\bar{x}}_{v+1} + b \ddot{\bar{x}}_{v+2}$$

where $a = (d_{v+1} - d_{v+2})/d_{v+1}$, $b = d_{v+2}/d_{v+1}$

3. The vectors \bar{s}_{λ} , $\bar{\omega}_{\lambda}$, and \bar{x} 's are related as follows:

$$\bar{s}_{\lambda} = (\bar{x}_{\lambda+1} - \bar{x}_{\lambda})d_{\lambda}, \quad \dot{\bar{s}}_{\lambda} = (\dot{\bar{x}}_{\lambda+1} - \dot{\bar{x}}_{\lambda})d_{\lambda}, \quad \ddot{\bar{s}}_{\lambda} = (\ddot{\bar{x}}_{\lambda+1} - \ddot{\bar{x}}_{\lambda})d_{\lambda}$$

$$\bar{s}_{\lambda} \cdot \dot{\bar{s}}_{\lambda} = -\dot{\bar{s}}_{\lambda}^2, \quad \bar{\omega}_{\lambda} = \bar{s}_{\lambda} \times \dot{\bar{s}}_{\lambda}, \quad \dot{\bar{\omega}}_{\lambda} = \bar{s}_{\lambda} \times \ddot{\bar{s}}_{\lambda},$$

$$\bar{q}_{\lambda} = -\bar{s}_{\lambda} \times (\bar{u}_{\lambda} \times \bar{s}_{\lambda}) / |\bar{u}_{\lambda} \times \bar{s}_{\lambda}|$$

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In this appendix, the linear and angular moments and their derivatives of the λ th element are considered.

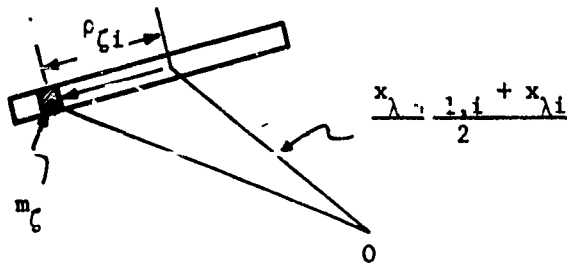


FIGURE B1. CABLE ELEMENT OF MASS m_ζ

Consider a small element of mass m_ζ as shown in Figure B1. Since ρ_i originates from the c.m. of the cable element, it follows from the definition of center of mass of bodies that

$$\sum m_\zeta \rho_{\zeta i} = 0, \text{ and } \sum m_\zeta \dot{\rho}_{\zeta i} = 0$$

Let $\omega_{\lambda i}$ be the components of angular velocity of the λ th cable element; then the velocity and acceleration of a typical small element of mass m_ζ are:

$$\begin{aligned} \text{vel of } m_\zeta &= \frac{(\dot{x}_{\lambda+1,i} + \dot{x}_{\lambda,i})}{2} + e_{ijk} \omega_{\lambda j} \rho_{\zeta k} \\ \text{acc of } m_\zeta &= \frac{(\ddot{x}_{\lambda+1,i} + \ddot{x}_{\lambda,i})}{2} + e_{ijk} \dot{\omega}_{\lambda j} \rho_{\zeta k} + e_{ijk} \omega_{\lambda j} \dot{\rho}_{\zeta k} \end{aligned}$$

In the above expressions, the summation convention has been used. Within one term, no small latin index is to be repeated more than once. When it is repeated once within a term a summation of this index from 1 to 3 is implicitly understood. If a small latin index is not repeated the equation is to hold for that index having each of the values 1, 2, 3.

The convention does not apply to Greek indices, and may be suspended by explicitly stating so. Further, e_{ijk} is the conventional three-index symbol, where $i = 1, 2$, or 3 , and j and k have the same range. If any two indices are equal, then $e_{ijk} = 0$. If i, j, k are an even permutation of $1, 2, 3$ the symbol has the value of $+1$. If i, j, k are an odd permutation, its value is -1 . For example, the i th component of $a \times b$ is $e_{ijk} a_j b_k$.

The rate of change of linear momentum of the λ th cable element is:

$$\begin{aligned} d[(\text{Linear momentum})_{\lambda i}] / dt &= d \left[\sum m_{\zeta} (\text{velocity})_{\zeta i} \right] / dt \\ &= \sum m_{\zeta} \frac{\ddot{x}_{\lambda+1, i} + \ddot{x}_{\lambda i}}{2} + e_{ijk} \dot{\omega}_{\lambda j} \underbrace{\sum m_{\zeta} \rho_{\zeta k}}_0 + e_{ijk} \omega_{\lambda j} \underbrace{\sum m_{\zeta} \dot{\rho}_{\zeta k}}_0 \end{aligned}$$

where the sums are carried out as ζ ranges over the λ th element. The above expression reduces to the following:

$$d[(\text{Linear momentum})_{\lambda i}] / dt = (\text{mass}_{\lambda}) \frac{(\ddot{x}_{\lambda+1, i} + \ddot{x}_{\lambda i})}{2}$$

Let $H_{\lambda i}$ be the angular momentum about the c.g., then the rate of change of the angular momentum of the λ th cable element is:

$$d(H_{\lambda i}) / dt = \sum m_{\zeta} e_{ijk} \rho_{\zeta j} (\text{acc})_{\zeta k} = \dot{H}_{\lambda i}$$

or

$$\begin{aligned} \dot{H}_{\lambda i} &= e_{ijk} \frac{\ddot{x}_{\lambda+1, k} + \ddot{x}_{\lambda k}}{2} \underbrace{\sum m_{\zeta} \rho_{\zeta j}}_0 + e_{ijk} \sum m_{\zeta} \rho_{\zeta j} e_{kpq} \dot{\omega}_{\lambda p} \rho_{\zeta q} \\ &+ e_{ijk} \sum m_{\zeta} \rho_{\zeta j} e_{kpq} \omega_{\lambda p} (e_{qst} \omega_{\lambda s} \rho_{\zeta t}), \end{aligned}$$

in which $\rho_{\zeta q} = e_{qst} \omega_{\lambda s} \rho_{\zeta k}$, and $(\text{acc})_k$ derived earlier in this section have been substituted. For further simplification we recall the following relation (Reference 5):

$$e_{ijk} e_{kpq} = e_{kij} e_{kpq} = \delta_p^i \delta_q^j - \delta_q^i \delta_p^j$$

so that the second term in the above expression for $\dot{H}_{\lambda i}$ may be written as follows:

$$\begin{aligned} e_{ijk} \sum m_{\zeta}^{\rho} \zeta_j e_{kpq} \dot{\omega}_{\lambda p} \rho_{\zeta q} &= (\delta_p^i \delta_q^j - \delta_q^i \delta_p^j) \sum m_{\zeta}^{\rho} \zeta_j \rho_{\zeta q} \dot{\omega}_{\lambda p} \\ &= \dot{\omega}_{\lambda i} \sum m_{\zeta}^{\rho} \zeta_j \rho_{\zeta j} - \dot{\omega}_{\lambda j} \sum m_{\zeta}^{\rho} \zeta_i \rho_{\zeta j} \\ &= \dot{\omega}_{\lambda i} I_{jj}^{\lambda} - \dot{\omega}_{\lambda j} I_{ij}^{\lambda}, \end{aligned}$$

where $I_{jj}^{\lambda} = \sum m_{\zeta}^{\rho} \zeta_j \rho_{\zeta j}$ and $I_{ij}^{\lambda} = \sum m_{\zeta}^{\rho} \zeta_i \rho_{\zeta j}$

The contribution of the last term in $\dot{H}_{\lambda i}$ may be rearranged into the following form, where ζ ranges over the λ th element and is indicated by the summation symbol:

$$e_{ijk} e_{kpq} e_{qst} \sum m_{\zeta}^{\rho} \zeta_j \rho_{\zeta t} \omega_{\lambda p} \omega_{\lambda s},$$

which reduces to the following form:

$$\begin{aligned} (\delta_p^i \delta_q^j - \delta_q^i \delta_p^j) e_{qst} \omega_{\lambda p} \omega_{\lambda s} \sum m_{\zeta}^{\rho} \zeta_j \rho_{\zeta t} \\ = e_{\lambda i \lambda s} e_{jst} I_{jt}^{\lambda} - e_{ist} \omega_{\lambda j} \omega_{\lambda s} I_{jt}^{\lambda} \end{aligned}$$

where as before $I_{jt}^{\lambda} = \sum m_{\zeta}^{\rho} \zeta_j \rho_{\zeta t}$. Moreover the symmetry of I_{jt}^{λ}

in j and t implies that the first term of this sum is zero, which is proved as follows:

$$\begin{aligned} 2e_{jst} \omega_{\lambda i} \omega_{\lambda s} I_{jt}^{\lambda} &= e_{jst} \omega_{\lambda i} I_{jt}^{\lambda} + e_{jst} \omega_{\lambda i} \omega_{\lambda s} I_{tj}^{\lambda} \\ &= \omega_{\lambda i} \omega_{\lambda s} I_{jt}^{\lambda} (e_{jst} - e_{jst}) = 0 \end{aligned}$$

Finally the expression for $\dot{H}_{\lambda i}$ for the λ th cable element becomes:

$$\dot{H}_{\lambda i} = \dot{\omega}_{\lambda i} I_{jj}^{\lambda} - \dot{\omega}_{\lambda j} I_{ji}^{\lambda} - e_{ist} \omega_{\lambda j} \omega_{\lambda s} I_{jt}^{\lambda}$$

where I_{jt}^{λ} is summed over the λ th element. This expression for $\dot{H}_{\lambda i}$

holds whenever the expression used for $(acc)_{\zeta i}$ gives the acceleration of the ζ th particle element in the inertial frame. Since the latter is a vector, it is valid both in the inertial and in a frame fixed in the body. However, $\dot{H}_{\lambda i}$ enters into the equations of motion in moving coordinates directly only when the origin of the moving frame is at c.g.

To apply the \dot{H} expression to the vehicle, it is desirable to express I_{ij}^{v+1} as a time independent quantity. With this in mind, we imagine a frame fixed in the vehicle at its c.g., consequently the expression $\dot{H}_{v+1,n}$ with reference to this frame is

$$\dot{H}'_{v+1,n} = \dot{\omega}'_{v+1,n} I'_{jj}{}^{v+1} - \dot{\omega}'_{v+1,j} I'_{jn}{}^{v+1} - e_{nst} \omega'_{v+1,j} \omega'_{v+1,s} I'_{jt}{}^{v+1}$$

However

$$\dot{H}'_{v+1,n}, \quad \omega'_{v+1,n}, \quad \dot{\omega}'_{v+1,n} \quad \text{are vectors, so that}$$

$$\dot{H}_{v+1,i} = c_{in} \dot{H}'_{v+1,n}, \quad \dot{\omega}_{v+1,i} = c_{in} \dot{\omega}'_{v+1,n}$$

$$\omega'_{v+1,j} = \omega_{v+1,\ell} c_{\ell j}, \quad \omega'_{v+1,j} = \omega_{v+1,\ell} c_{\ell j}, \quad \omega'_{v+1,s} = \omega_{v+1,k} c_{ks}$$

where the c_{ij} are the coefficients found in Appendix G.

Finally a substitution of the above quantities into the expression for $\dot{H}_{v+1,i}$ gives:

$$\begin{aligned} \dot{H}_{v+1,i} = & \dot{\omega}_{v+1,i} I_{jj}{}^{v+1} - \dot{\omega}_{v+1,\ell} c_{in} c_{\ell j} I_{jn}{}^{v+1} \\ & - e_{nst} \omega_{v+1,\ell} \omega_{v+1,k} c_{in} c_{\ell j} c_{ks} I_{jt}{}^{v+1} \end{aligned}$$

where the \dot{H} , the ω 's, and $\dot{\omega}$'s are relative to the inertial frame and the I 's are calculated relative to the vehicle fixed frame of reference.

The expression $\dot{H}_{\lambda i}$ for a cable element, however, may be expressed with reference to the inertial frame. The moments of inertia with respect to the inertial frame are: (where $A = m_{\lambda} d^2/12$)

$$I_{11}^{\lambda} = A s_x s_x, \quad I_{12}^{\lambda} = A s_x s_y, \quad I_{13}^{\lambda} = A s_x s_z$$

$$I_{22}^{\lambda} = A s_y s_y, \quad I_{23}^{\lambda} = A s_y s_z, \quad I_{33}^{\lambda} = A s_z s_z$$

where s_x, s_y, s_z are components of the unit vector tangent to the cable vector, and these components are known at any instant of time. Here the cable diameter d is assumed to be made much less than the length of a cable element.

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APPENDIX C

GEOMETRICAL FORCES

There are two types of geometrical forces on each cable element. They are the weight of cable in water, and the force exerted by adjoining elements. The vehicle possesses these forces also, and in addition has a moment about the c.g. due to the displacement of the center of buoyancy from the c.g.

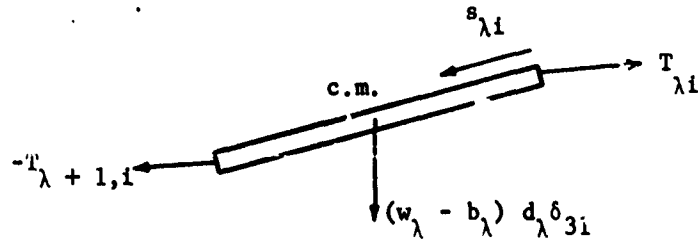


FIGURE C1. CABLE TENSIONS AND NET WEIGHT

Let $\bar{s}_{\lambda i}$ be the unit tangent vector, w_{λ} the weight per unit length of cable in air, and b_{λ} the buoyant force per unit length of cable.

The force on a cable element is shown in Figure C1:

$$F_{G\lambda i} = T_{\lambda i} - T_{\lambda+1,i} + (w_{\lambda} - b_{\lambda}) d_{\lambda} \delta_{3i}$$

and the moment of these forces about the c.m. of the element is:

$$\begin{aligned} M_{G\lambda i} &= (d_{\lambda}/2) [-e_{ijk} s_{\lambda j} T_{\lambda k} + e_{ijk} s_{\lambda j} (-T_{\lambda+1,k})] \\ &= -\frac{1}{2} e_{ijk} d_{\lambda} s_{\lambda j} (T_{\lambda k} - T_{\lambda+1,k}) \end{aligned}$$

The force and moment on the vehicle are:

$$F_{Gv+1,i} = T_{v+1,i} - T_{v+2,i} + (w_{v+1} - b_{v+1}) d_{v+1} \delta_{3i}$$

$$M_{Gv+1,i} = e_{ijk} s_{v+1,j} [d_{v+2} - d_{v+1}] T_{v+2,k} - d_{v+2} e_{ijk} s_{v+1,j} T_{v+1,k} \\ - e_{ijk} s_{v+1,j} \delta_{3k} (d_{v+1} - d_{v+2}) b_{v+1} d_{v+1}$$

where $b_{v+1} = (\text{buoyant force on vehicle})/d_{v+1}$. Note that the force applied to the stern of the vehicle is $(-T_{v+2,i})$.

APPENDIX D

VELOCITY FORCES

There are two forces dependent on the velocity; the tangential and normal forces.

We assume as in Reference 2 that

1. The tangential force has the directions \bar{s}_λ .
2. The magnitude of the tangential force does not depend on the orientation of element.
3. The magnitude of the tangential force is $(1/2)\rho\bar{u}_\lambda^2 A(\text{constant}) = C_1 \bar{u}_\lambda^2 d_\lambda$, where \bar{u}_λ is the velocity through the water.
4. The normal force is in the direction \bar{q}_λ , that is in the plane of the concurrent vectors \bar{s}_λ and \bar{u}_λ and normal to \bar{s}_λ .
5. The magnitude of the normal force is the product of two factors $\chi(\bar{u}_\lambda^2)\psi(\varphi_\lambda)$, where φ_λ is the angle between \bar{s}_λ and \bar{u}_λ .

In addition

$$\psi(\varphi_\lambda) = \sin^2 \varphi_\lambda ,$$

$$\chi(\bar{u}_\lambda^2) = (1/2)\rho\bar{u}_\lambda^2 A(\text{constant}) = C_2 \bar{u}_\lambda^2 d_\lambda$$

The \bar{u}_λ appearing above varies along the element due to its rotation. We ask what error is committed using the \bar{u}_λ of the midpoint of the cable element.

Let ΔF_λ be the magnitude of some force due to the velocity through the water of $\Delta\rho$, as shown in Figure D1.

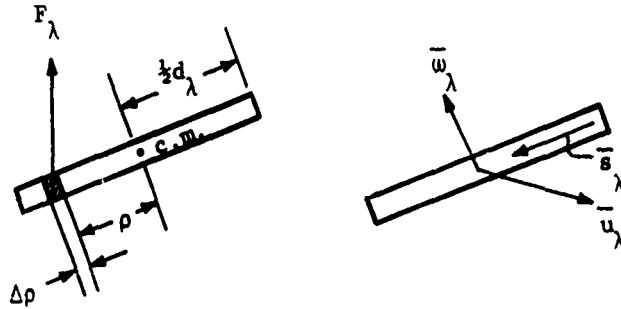


FIGURE D1. FORCE DUE TO VELOCITY

The incremental force on an element of length $\Delta\rho$ is:

$$\Delta F_{\lambda} = K(\bar{u}_{\lambda} + \bar{\omega}_{\lambda} \times \bar{\rho})^2 \Delta\rho$$

The integral of the incremental forces is as follows:

$$(F_{\text{acc}})_{\lambda} = K \int_{d_{\lambda}/2}^{d_{\lambda}/2} \left[\bar{u}_{\lambda}^2 + 2\bar{u}_{\lambda} \cdot (\bar{\omega}_{\lambda} \times \bar{\rho}) + (\bar{\omega}_{\lambda} \times \bar{\rho})^2 \right] d\rho$$

The above integral may be simplified if we observe that:

$$2\bar{u}_{\lambda} \cdot (\bar{\omega}_{\lambda} \times \bar{\rho}) = 2\bar{u}_{\lambda} \cdot (\bar{\omega}_{\lambda} \times \bar{s}_{\lambda} \rho) = f(u_{\lambda}, \omega_{\lambda}) \rho$$

$$(\bar{\omega}_{\lambda} \times \bar{\rho})^2 = \omega_{\lambda}^2 \rho^2 \sin^2(\bar{\rho}, \bar{\omega}_{\lambda}) = \omega_{\lambda}^2 \rho^2,$$

since by hypothesis $\sin(\bar{\rho}, \bar{\omega}_{\lambda}) = \pm 1$. Here $\bar{\omega}_{\lambda}$ is taken to be a vector normal to the vector $\bar{\rho}$.

Also noting that

$$\int_{-d_\lambda/2}^{d_\lambda/2} \rho dp = 0, \text{ and } \int_{-d_\lambda/2}^{+d_\lambda/2} \rho^2 dp = d_\lambda^3/12,$$

then if $(F_{app})_\lambda$ be the approximate force obtained by neglecting the angular velocity $\bar{\omega}_\lambda$, we have

$$(F_{app})_\lambda = K \int_{-d_\lambda/2}^{+d_\lambda/2} u_\lambda^2 dp = Ku_\lambda^2 d_\lambda$$

The relative error is

$$\frac{(F_{acc})_\lambda - (F_{app})_\lambda}{(F_{app})_\lambda} = (1/12) (\omega_\lambda d_\lambda / u_\lambda)^2$$

so that if the velocity due to rotation of one end of the element with respect to the other were equal to the velocity of the midpoint through the water, the omission of rotation causes an error of 8.25 per cent. If the relative velocity due to rotation were one-half of the translation velocity, the error would be about 2 per cent. This can always be achieved by using short cable elements, but apparently this is not a stringent limitation. Therefore when calculating the velocity forces, the rotation of the element will be ignored.

By a direct three dimension generalization of the normal and tangential forces discussed in Reference 2, one can obtain a representation of the tangential force with a magnitude of $d_\lambda u_\lambda^2 C_1$ and direction \bar{s}_λ ; and a representation of the normal force with a magnitude of

$$d_\lambda C_2 u_\lambda^2 \sin^2 \varphi_\lambda = d_\lambda C_2 u_\lambda^2 (-\bar{s}_\lambda \times \bar{u}_\lambda / |u_\lambda|)^2 = d_\lambda C_2 (\bar{s}_\lambda \times \bar{u}_\lambda)^2$$

with a direction of $\bar{q}_{\lambda 1}$.

The moment above the c.m. due to lift is now computed. Let \bar{v} be the velocity of $\Delta\rho$ through the water, then $\bar{v} = \bar{u}_\lambda + \bar{\omega}_\lambda \times \rho \bar{s}_\lambda$. The incremented force experienced by the element becomes:

$$\Delta F = C_2 \bar{v}^2 \sin^2 (\bar{v}, -\bar{s}_\lambda) \Delta \rho$$

where $\bar{v} \times (-\bar{s}_\lambda) = \bar{v} \sin (\bar{v}, -\bar{s}_\lambda)$. Then ΔF takes the form

$\Delta F = C_2 |\bar{s} \times \bar{v}|^2 \Delta \rho$. The element moment about the c.m. of the cable element takes the form:

$$\Delta \bar{M} = \rho \bar{s}_\lambda \times (\Delta F) \bar{q}_\lambda = \rho C_2 |\bar{s}_\lambda \times \bar{v}|^2 \bar{s}_\lambda \times \bar{q}_\lambda \Delta \rho$$

since $\bar{s}_\lambda \times \bar{v}^2 = s_\lambda^2 v^2 - (\bar{s}_\lambda \cdot \bar{v})^2$, then an integration from $-d_\lambda/2$ to $d_\lambda/2$ yields:

$$\bar{M}_{V\lambda} = C_2 (\bar{s}_\lambda \times \bar{q}_\lambda) \int_{-d_\lambda/2}^{d_\lambda/2} [\rho \bar{v}^2 - (\bar{s}_\lambda \cdot \bar{v})^2] d\rho$$

Noting that $(\bar{s}_\lambda \cdot \bar{v})^2 = (\bar{s}_\lambda \cdot \bar{u}_\lambda)^2$, and also that integration of terms with odd powers of ρ will give no net contributions, the above expression becomes:

$$\begin{aligned} \bar{M}_{V\lambda} &= C_2 \bar{s}_\lambda \times \bar{q}_\lambda \int_{-d_\lambda/2}^{d_\lambda/2} \\ &= C_2 \bar{s}_\lambda \times \bar{q}_\lambda (\bar{u}_\lambda \cdot \bar{\omega}_\lambda \times \bar{s}_\lambda) \int_{-d_\lambda/2}^{+d_\lambda/2} 2\rho^2 d\rho \end{aligned}$$

Finally the moment of the lift about the c.m. reduces to the form

$$\bar{M}_{V\lambda} = (C_2 d_\lambda^3/6) (\bar{u}_\lambda \cdot \bar{\omega}_\lambda \times \bar{s}_\lambda) \bar{s}_\lambda \times \bar{q}_\lambda$$

APPENDIX E

ACCELERATION FORCES

In this appendix there are two mutually exclusive but alternative physical assumptions to be made regarding the composition of these forces. The first, which we label Approach A, is a three-dimensional formulation of the physical assumption embodied in Reference 1. The second, which we label Approach B, is a Kirchhoff's general formulation of the added masses of a submerged body.

The mathematical expression of Approach A is simpler than that of Approach B. Approach A is always used for the cable. Approach B may be more desirable for the vehicle. Equations are derived for each approach.

These forces and moments contain the accelerations. The accelerations are the unknowns for which the final linear equations are to be solved. Approach A and Approach B assign different values to the coefficients of these equations, and therefore differ basically.

APPROACH A

In Approach A we are concerned only with the force due to the induced acceleration of the fluid. As in Reference 1 we assume that this force is proportional to the component of the acceleration normal to the cable element, and that there is no moment due to the acceleration.

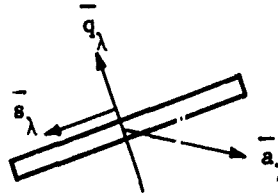


FIGURE E1. UNIT VECTORS \bar{s}_λ , \bar{q}_λ , AND ACCELERATION \bar{a}_λ

Let \bar{n}_λ be the unit vector normal to \bar{s}_λ and in the plane of \bar{a}_λ and \bar{s}_λ , as shown in Figure E1. The acceleration \bar{a}_λ is

$$\bar{a}_\lambda = (\bar{a}_\lambda \cdot \bar{n}_\lambda) \bar{n}_\lambda + (\bar{a}_\lambda \cdot \bar{s}_\lambda) \bar{s}_\lambda$$

Then the component of acceleration normal to the cable element is:

$$(\bar{a}_\lambda \cdot \bar{n}_\lambda) \bar{n}_\lambda = \bar{a}_\lambda - (\bar{a}_\lambda \cdot \bar{s}_\lambda) \bar{s}_\lambda,$$

Hence the force due to acceleration of the element is

$$\begin{aligned} \bar{F}_{A\lambda} &= (-e'm_\lambda) \bar{a}_\lambda \cdot \bar{n}_\lambda = -e'm(\bar{a}_\lambda - \bar{a}_\lambda \cdot \bar{s}_\lambda \bar{s}_\lambda) \\ &= -e_\lambda (\bar{a}_\lambda - \bar{a}_\lambda \cdot \bar{s}_\lambda \bar{s}_\lambda) \end{aligned}$$

where $e'm_\lambda = e_\lambda$ is a parameter which depends upon the added mass of fluid of cable element.

APPROACH B

Consider a body moving in a perfect fluid which is otherwise at rest. The motion of the fluid is assumed to be irrotational. The hydrodynamic forces and moments experienced by this body due to its linear and angular accelerations are given by the following (Reference 3):

$$\begin{aligned} F_i &= -\rho \left[\sum_{j=1}^3 \dot{u}_j^0(t) u_{ij} + \sum_{j=1}^3 \dot{\omega}_j(t) \mu_{i,j+3} \right], \quad (i=1,2,3) \\ M_i &= -\rho \left[\sum_{j=1}^3 \dot{u}_j^0(t) \mu_{i+3,j} + \sum_{j=1}^3 \dot{\omega}_j(t) \mu_{i+3,j+3} \right], \quad (i=1,2,3) \end{aligned}$$

$u_j^0(t)$ are the components of the translational velocity of the origin of a system of rectangular axes fixed to the body, and ω_j are the components of angular velocity of the body about this origin as shown in Figure E2. The density of the fluid is given by ρ , and μ_{ij} are the "added masses". The constants μ_{ij} are determined exclusively by the shape and position of the body surface relative to the moving coordinate axes fixed to the body. It is also known that $\mu_{ij} = \mu_{ji}$, consequently the above equations for forces and moments involve 21 possible independent constants (Reference 3). If the surface of the body has one or more planes of symmetry, the number of independent constants are reduced. For example, if the $\xi_1 \xi_2$ - plane is a plane of symmetry, it can be proven that 21 independent constants reduce to 12. If the $\xi_2 \xi_3$ - plane is also a plane of symmetry, then the 12 independent constants reduce to 8. If there is also symmetry of body surface about the $\xi_1 \xi_3$ - plane, the

8 independent constants reduce to 6, so that only the constants μ_{ij} ($i = 1, \dots, 6$) remain. However, in this analysis we shall retain all the constants or "added masses."

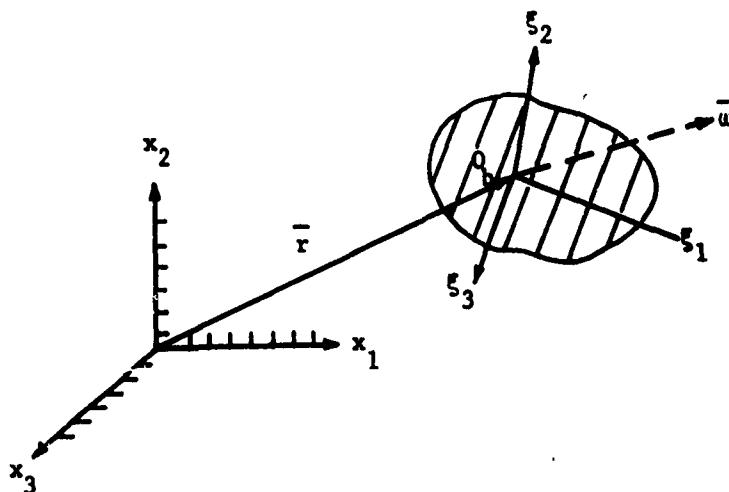


FIGURE E2. FIXED FRAME x_1, x_2, x_3 , AND BODY FIXED FRAME ξ_1, ξ_2, ξ_3

Let $\ddot{\xi}_1, \ddot{\xi}_2, \ddot{\xi}_3$ denote the components of linear acceleration of the moving origin O_b in the direction of the axes fixed to the body. Also let F_{ξ_i} and M_{ξ_i} denote the components of hydrodynamic forces and moments respectively in the direction of the axes fixed to the body. The above equations for the forces and moments now take the following form:

$$F_{\xi_i} = -\rho \left[C1_{ij} \ddot{\xi}_j - C3_{ij} \dot{\omega} \xi_j \right]$$

$$M_{\xi_i} = -\rho \left[C3_{ij} \ddot{\xi}_j - C2_{ij} \dot{\omega} \xi_j \right]$$

where $C1_{ij} = \mu_{ij}$ for $i \leq 3, j \leq 3$; $C2_{ij} = \mu_{1+3, j+3}$, for $i + 3 > 3, j + 3 > 3$; $C3_{ij} = \mu_{i, j+3}$, for $i \leq 3, j + 3 > 3$, in the equation for the force; and $C3_{ij} = \mu_{i+3, j}$, $i + 3 > 3, j \leq 3$, in the equation for the moment.

To refer the forces and moments relative to the directions of the inertial axes we note that for any vector V_i we can write $V_{x_i} = c_{ij} V_{\xi_j}$, i.e., $V_i = c_{ij} V'_j$, and $V'_k = c_{kj} V_j$. Here the prime indicates evaluation with respect to the moving frame, as shown in Figure E3.

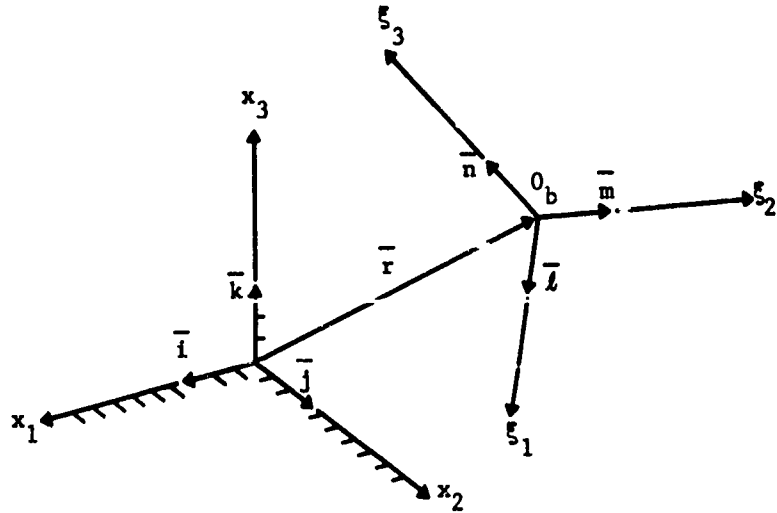


FIGURE E3. UNIT VECTORS ON FIXED AND MOVING FRAME

Referring \bar{V} to the basis $\bar{i}, \bar{j}, \bar{k}$ and \bar{V}' to the basis $\bar{l}, \bar{m}, \bar{n}$, then $d\bar{V}/dt = d\bar{V}'/dt + \bar{\omega} \times \bar{V}$. In particular $d\bar{\omega}/dt = d\bar{\omega}'/dt + \bar{\omega} \times \bar{\omega} = d\bar{\omega}'/dt$, so that $\omega_{x_i} = c_{ij} \omega_{\xi_j}$.

Hence

$$\begin{aligned} F_{x_i} &= c_{ij} \bar{F}_{\xi_j} = -\rho c_{ij} [C1_{ij} \xi_k - C3_{jk} \dot{\omega}_{\xi_k}] \\ &= -\rho c_{ij} [C1_{jk} \xi_k - C3_{jk} c_{lk} \dot{\omega}_{x_l}] \end{aligned}$$

where $\dot{\omega}_{\xi_k} = c_{lk} \dot{\omega}_{x_l}$, and

$$M_{x_i} = c_{ij} \bar{M}_{\xi_j} = -\rho c_{ij} [C3_{jk} \xi_k - C2_{jk} c_{lk} \dot{\omega}_{x_l}]$$

The next step in the analysis is concerned with the expression for ξ_k in terms of the inertial coordinates x_1, x_2, x_3 . From Figure E3, and

also from a coordinate transformation we write $x_i = c_{ij} \xi_j + d_i$, where d_1, d_2, d_3 are the x_1, x_2, x_3 coordinates of the origin of the moving frame. The transformation is also orthogonal so that $c_{ij} c_{ik} = \delta_{jk}$, $c_{ij} c_{kj} = \delta_{ik}$, and $\xi_j = c_{ij} (x_i - d_i)$. We also note that

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \xi\vec{\ell} + \eta\vec{m} + \zeta\vec{n},$$

and then by the usual rules of kinematics we obtain:

$$\dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = \dot{\xi}\vec{\ell} + \dot{\eta}\vec{m} + \dot{\zeta}\vec{n} + \vec{\omega} \times \vec{r},$$

$$\begin{aligned} \text{and} \quad \ddot{\vec{r}} &= \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} \\ &= \ddot{\xi}\vec{\ell} + \ddot{\eta}\vec{m} + \ddot{\zeta}\vec{n} + 2\vec{\omega} \times (\dot{\xi}\vec{\ell} + \dot{\eta}\vec{m} + \dot{\zeta}\vec{n}) + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \end{aligned}$$

This equation can be rearranged in the following form:

$$\begin{aligned} \ddot{\xi} &= \ddot{\xi}\vec{\ell} + \ddot{\eta}\vec{m} + \ddot{\zeta}\vec{n} = \ddot{\vec{r}} - 2\vec{\omega} \times \dot{\vec{r}} - (\vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= \overline{AA} + \overline{AB} \end{aligned}$$

Where $\overline{AA} = \ddot{\vec{r}} - \vec{\omega} \times \vec{r}$, so that $AA_j = \ddot{x}_j - e_{j pq} \dot{\omega}_p x_q$;

and $\overline{AB} = -2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$, so that $AB_j = -e_{j pq} 2\dot{\omega}_p \dot{x}_q$

+ $e_{j pq} \omega_p e_{qst} \omega_s x_t$. Thus the k th component of $\ddot{\xi}$ is:

$$\begin{aligned} \ddot{\xi}_k &= c_{jk} (AA_j + AB_j) \\ &= c_{jk} (\ddot{x}_j - e_{j pq} \dot{\omega}_p x_q) + c_{jk} (e_{j pq} e_{qst} \omega_p \omega_s x_t - 2e_{j pq} \dot{\omega}_p \dot{x}_q) \end{aligned}$$

where the first term on the right hand side contains accelerations, the second term does not contain accelerations, and x_q is the q th component of \vec{r} where \vec{r} is the radius vector from the inertial origin to the origin of the moving frame. Finally

$$\begin{aligned} F_{x_i} &= -\rho c_{ij} C1_{jk} c_{rk} (\ddot{x}_r - e_{rpq} \dot{\omega}_p x_q) - \rho c_{ij} C3_{jk} c_{lk} \dot{\omega}_l x_k \\ &\quad - \rho c_{ij} C1_{jk} c_{rk} (e_{j pq} e_{qst} \omega_p \omega_s x_t - 2e_{j pq} \dot{\omega}_p \dot{x}_q) \\ M_{x_i} &= -\rho c_{ij} C3_{jk} c_{rk} (\ddot{x}_r - e_{rpq} \dot{\omega}_p x_q) - \rho c_{ij} C2_{jk} c_{lk} \dot{\omega}_l x_k \\ &\quad - \rho c_{ij} C3_{jk} c_{rk} (e_{j pq} e_{qst} \omega_p \omega_s x_t - 2e_{j pq} \dot{\omega}_p \dot{x}_q) \end{aligned}$$

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APPENDIX F

FIN FORCES AND MOMENTS

The vehicle considered here has a pair of fins in a vertical plane (one fin above the vehicle and one below), and a pair of fins in the horizontal plane (one to port and one to starboard). These fins are subsequently referred to as the vertical fins and the horizontal fins, respectively.

Consider first the vertical fins. Figure F1 illustrates the forward half of one of the vertical fins. The velocity through the fluid of the vertical fins (see Appendix A) is:

$$\bar{v}_{vf} = \dot{x}_{v+1} + (d_{v+4}/d_{v+1}) (\dot{x}_{v+2} - \dot{x}_{v+1})$$

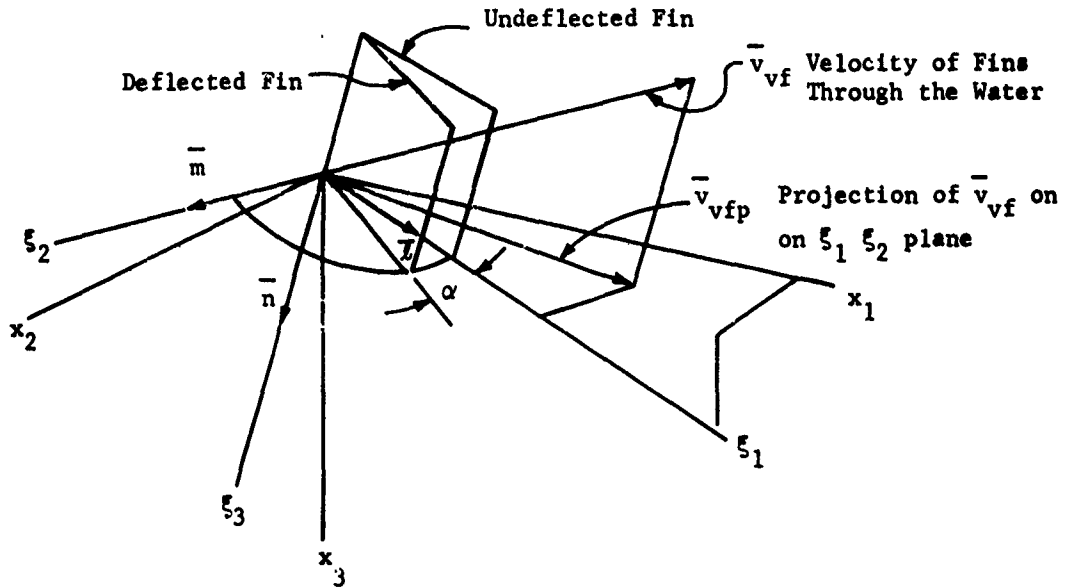


FIGURE F1. VERTICAL FIN AND ITS VELOCITY

The lift (or sidewise force in the case of vertical fins) will be determined by the general relation $\bar{L} = \bar{\Gamma} \times \rho \bar{U}$, where \bar{L} is the lift, $\bar{\Gamma}$ is the circulation, and \bar{U} is the velocity of the fins. The vector $\bar{\Gamma}$ will be assumed to be equal to:

(constant)(velocity)(sine of the angle of attack)(unit vector). The effect of spanwise flow of the fluid will be ignored in this formulation. This assumption implies: (1) that the circulation vector is parallel to the axis of the fin, i.e. $\pm \bar{n}$; (2) that the angle of attack is measured in the $\xi_1 \xi_2$ plane, as shown in Figure F2; and (3) that the velocity with which we are concerned is $\bar{v}_{vf} - (\bar{v}_{vf} \cdot \bar{n})\bar{n}$, i.e. the projection of \bar{v}_{vf} on the $\xi_1 \xi_2$ plane.

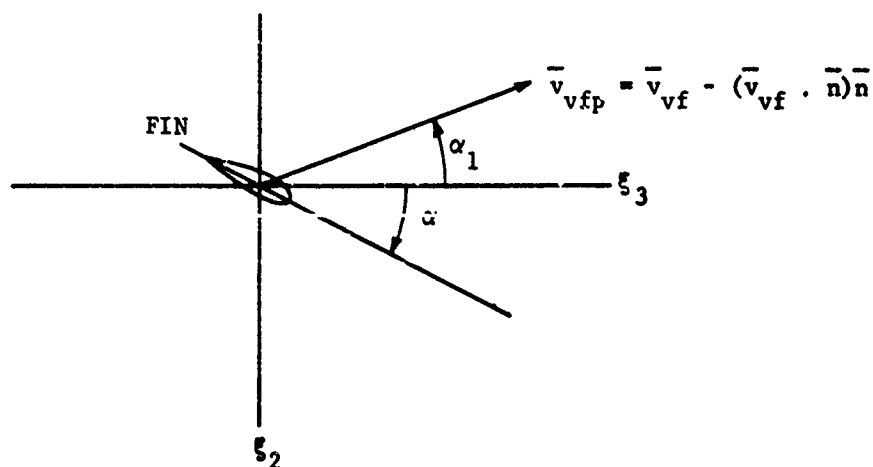


FIGURE F2. FIN AND VELOCITY IN $\xi_1 \xi_2$ PLANE

The angle of attack is $\alpha + (-\alpha_1)$, where α is the known deflection of the fin relative to the vehicle, where $\alpha_1 = \arctan (m \cdot v_{vfp} / l \cdot v_{vfp}) = \arctan (\bar{m} \cdot \bar{v}_{vf} / l \cdot \bar{v}_{vf})$. Hence the circulation vector becomes $\bar{\Gamma} = (\text{constant}) v_{vfp} \bar{n} \sin(\alpha - \alpha_1)$ and the sidewise force takes the form:

$$(\text{constant}) v_{vfp} \sin(\alpha - \alpha_1) \bar{n} \times \rho (\bar{v}_{vf} - (\bar{v}_{vf} \cdot \bar{n})\bar{n})$$

where v_{vfp} is the magnitude of \bar{v}_{vfp} . Let \bar{q}_{v+2} be a unit vector in the direction \bar{v}_{vf} , and \bar{q}_{v+3} be another unit vector defined by $\bar{n} \times \bar{q}_{v+2}$. Since $\bar{n} \times \bar{n} = 0$, the vector factor in the above equation $\bar{n} \times \rho \bar{v}_{vf} = \rho v_{vf} \bar{q}_{v+3} \sin(\bar{n}, \bar{v}_{vf}) = \rho v_{vf} \bar{q}_{v+3}$. Then the lift = (constant) $\rho v_{vfp}^2 \bar{q}_{v+3} \sin(\alpha - \alpha_1)$. Since the fin is of low aspect ratio and is in the neighborhood of a large body, the lift may be written in the following form:

$$C_3 F_{LVF}(\alpha - \alpha_1) v_{vfp}^2 \bar{q}_{v+3}$$

where C_3 is an empirical constant greater than zero, and $F_{LVF}(\alpha - \alpha_1)$ is an empirical odd function of its argument, the angle of attack. Both C_3 and F_{LVF} may be obtained experimentally.

If α is positive when the forward edge of the fin is displaced to starboard, and F_{LVF} is positive when its argument is positive, then positive lift (sidewise force) tends to force the vehicle to starboard and to increase the $x_{v+1,2}$ coordinate. The drag on the vehicle is similarly

$$-C_4 F_{DVF}(\alpha - \alpha_1) v_{vfp}^2 \bar{q}_{v+2}$$

where, however, F_{DVF} is an even function of its argument. The moment about the center of gravity of the vehicle due to the vertical fins is:

$$\begin{aligned} & (\bar{v}_{vfp})^2 F_{LVF}(\alpha - \alpha_1) C_3 (d_{v+4} - d_{v+2}) \bar{s}_{v+1} \times \bar{q}_{v+2} \\ & - (\bar{v}_{vfp})^2 C_4 \bar{s}_{v+1} \times \bar{q}_{v+2} F_{DVF}(\alpha - \alpha_1) (d_{v+4} - d_{v+2}) \end{aligned}$$

which reduces to the following form:

$$\begin{aligned} & (\bar{v}_{vfp})^2 (d_{v+4} - d_{v+2}) \bar{s}_{v+1} \times [C_3 \bar{q}_{v+3} F_{LVF}(\alpha - \alpha_1) - \\ & C_4 \bar{q}_{v+2} F_{DVF}(\alpha - \alpha_1)] \end{aligned}$$

By proceeding in an analogous manner, similar results are obtained for the horizontal fins. In this case the velocity of the horizontal fins is:

$$\bar{v}_{hf} = \dot{\bar{x}}_{v+1} + (d_{v+3}/d_{v+1}) (\dot{\bar{x}}_{v+2} - \dot{\bar{x}}_{v+1})$$

Let \bar{q}_{v+4} be a unit vector in the direction of \bar{v}_{vf} ; $\bar{q}_{v+5} = -\bar{m} \times \bar{q}_{v+4}$; v_{hfp} be the projection of \bar{v}_{hf} on the $\xi_1 \xi_2$ -plane; $\beta_1 = \arctan(\bar{n} \cdot \bar{v}_{hf} / \bar{l} \cdot \bar{v}_{hf})$; and let β be positive when the leading edge of the fin is deflected down.

The lift due to the horizontal fins is then

$$C_5 (\bar{v}_{hfp})^2 \bar{q}_{v+5} F_{LHF}(\beta - \beta_1)$$

where C_6 and $F_{LHF}(\beta - \beta_1)$ are analogous to C_3 and F_{VHF} , respectively. Therefore $\beta > 0$ tends to increase $x_{v+1,3}$. The drag on the horizontal fins is $-C_6 (\bar{v}_{hfp})^2 \bar{q}_{v+4} F_{DHF}(\beta - \beta_1)$; and the moment about the center of gravity of these forces is:

$$(\bar{v}_{hfp})^2 (d_{v+3} - d_{v+1}) \bar{s}_{v+1} \times [C_5 \bar{q}_{v+5} F_{LHF}(\beta - \beta_1) - C_6 \bar{q}_{v+4} F_{DHF}(\beta - \beta_1)]$$

The total force \overline{FIN} , and moment \overline{FINM} acting on the vehicle are the sums of the above forces and the sums of the above moments. They are:

$$\begin{aligned} \overline{FIN} = & (\bar{v}_{vfp})^2 [C_3 \bar{q}_{v+3} F_{LVF}(\alpha - \alpha_1) - C_4 \bar{q}_{v+2} F_{DVF}(\alpha - \alpha_1)] \\ & + (\bar{v}_{hfp})^2 [C_5 \bar{q}_{v+5} F_{LHF}(\beta - \beta_1) - C_6 \bar{q}_{v+4} F_{DHF}(\beta - \beta_1)], \end{aligned}$$

and

$$\begin{aligned} \overline{FINM} = & (\bar{v}_{vfp})^2 (d_{v+4} - d_{v+2}) \bar{s}_{v+1} \times [C_3 \bar{q}_{v+3} F_{LVF}(\alpha - \alpha_1) \\ & - C_4 \bar{q}_{v+2} F_{DVF}(\alpha - \alpha_1)] \\ & + (\bar{v}_{hfp})^2 (d_{v+3} - d_{v+2}) \bar{s}_{v+1} \times [C_5 \bar{q}_{v+5} F_{LHF}(\beta - \beta_1) \\ & - C_6 \bar{q}_{v+4} F_{DHF}(\beta - \beta_1)] \end{aligned}$$

APPENDIX G

COORDINATE TRANSFORMATION BETWEEN INERTIAL COORDINATES AND COORDINATES FIXED IN THE VEHICLE WITH ORIGIN AT THE CENTER OF GRAVITY

The coordinates fixed in the vehicle with their origin at c.g. of vehicle are denoted by ξ, η, ζ . The coordinates x, y, z denoted the coordinates in a frame which is parallel to the inertial frame. The origin of both frames is at 0, and the ζ, η plane contains the horizontal fins, as shown in Figure G1. This appendix provides the coefficients c_{ij} to transform vector components between the two frames.

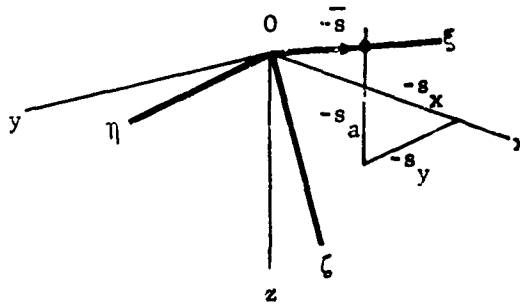


FIGURE G1. INERTIAL AND BODY-FIXED COORDINATES

For a linear orthogonal transformation $x_i = c_{ij}\xi_j$, (see Appendix B) the coefficients c_{ij} must satisfy the following orthonormal relations:

$$c_{ij}c_{ik} = \delta_{jk}, \quad c_{ji}c_{ki} = \delta_{kj}$$

The transformation in terms of matrices is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

If $\xi = 1$, $\eta = 0$, $\zeta = 0$, are the components of $-s$, then the substitution of these coordinates in the preceding matrix transformation gives

$$x = c_{11} \cdot (1.0) = -s_x$$

$$y = c_{21} \cdot (1.0) = -s_y$$

$$z = c_{31} \cdot (1.0) = -s_z$$

If next we let $\xi = 0$, $\eta = 1$, $\zeta = 0$, a substitution in the transformation with condition of zero roll gives $z = c_{32} \cdot (1.0) = 0$.

With the above mentioned quantities for x, y , and z , the coefficient transformation takes the following form:

$$\begin{pmatrix} -s_x & c_{12} & c_{13} \\ -s_y & c_{22} & c_{23} \\ -s_z & 0 & c_{33} \end{pmatrix}$$

The orthonormal relations are applied to find the relation between the c'_{ij} s and the s'_{ij} s. For example, $c_{31}c_{31} = 1 = s_z^2 + c_{33}^2$. Solving for

c_{33} we get $c_{33}^2 = 1 - s_z^2 = s_x^2 + s_y^2$, so that $c_{33} = \pm \sqrt{s_x^2 + s_y^2}$. when $s_z = 0$, $z = c_{33}\zeta$, so that we take the positive square root, hence

$$c_{33} = \sqrt{s_x^2 + s_y^2} = 1/\lambda$$

where λ is defined for convenience to be $1/\sqrt{s_x^2 + s_y^2}$.

For the second and third rows the orthonormal relations give

$$s_y s_z + c_{23} c_{33} = 0, \text{ thus } c_{23} = -s_y s_z / c_{33} = -s_y s_z \lambda.$$

For the first and third row the orthonormal relations give

$$s_x s_z + c_{13} c_{33} = 0, \text{ hence } c_{13} = -s_x s_z / c_{33} = -s_x s_z \lambda.$$

Also $c_{12}^2 + c_{22}^2 = 1$, and $-s_x c_{12} - s_y c_{22} = 0$, are consequences of the orthonormal relations. The last condition yields $s_x^2 c_{12}^2 = s_y^2 c_{22}^2$.

Using this relation and the other condition gives:

$$s_x^2 = s_x^2 c_{12}^2 + s_x^2 c_{22}^2 = s_y^2 c_{22}^2 + s_x^2 c_{22}^2$$

from which $c_{22}^2 = s_x^2 / (s_x^2 + s_y^2)$, hence $c_{22} = \mu \lambda s_x$, where μ denotes $+1$ or -1 . In addition

$$c_{12} = (-s_y / s_x) c_{22} = -\mu \lambda s_y$$

The determinant of the transformation is

$$\begin{vmatrix} -s_x & -\mu \lambda s_y & -s_x s_z \\ -s_y & \mu \lambda s_x & -s_y s_z \\ -s_z & 0 & (1/\lambda) \end{vmatrix}$$

which equals $-\mu s_z^2 - \mu (s_x^2 + s_y^2)$. Since the determinant must be equal to $+1$, then it follows that $\mu = -1$.

In summary, $x_i = c_{ij} \xi_j$, $\xi_j = c_{ij} x_i$, where

$$(c_{ij}) = \begin{pmatrix} -s_x & s_y / \sqrt{s_x^2 + s_y^2} & -s_x s_z / \sqrt{s_x^2 + s_y^2} \\ -s_y & -s_x / \sqrt{s_x^2 + s_y^2} & -s_y s_z / \sqrt{s_x^2 + s_y^2} \\ -s_z & 0 & \sqrt{s_x^2 + s_y^2} \end{pmatrix}$$

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This report presents the analysis of a system which includes a maneuvering ship towing an underwater vehicle at the end of a long flexible cable. The equations of motion for both the cable and the underwater vehicle are also presented.

The cable is imagined to consist of many interconnected short rigid segments. The equations of motion for the system are formulated twice on the basis of two hypotheses: first for a simple hypothesis regarding the inertia of an accelerated body in a fluid, and secondly, for a more complete and a more accurate hydrodynamic hypothesis.

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